Ultra-Relativistic Pistons

a new type of self-similarity

Tamar Faran (Princeton U), **Andrei Gruzinov** (NYU) & **Re'em Sari** (Hebrew U)

Self-Similar solutions

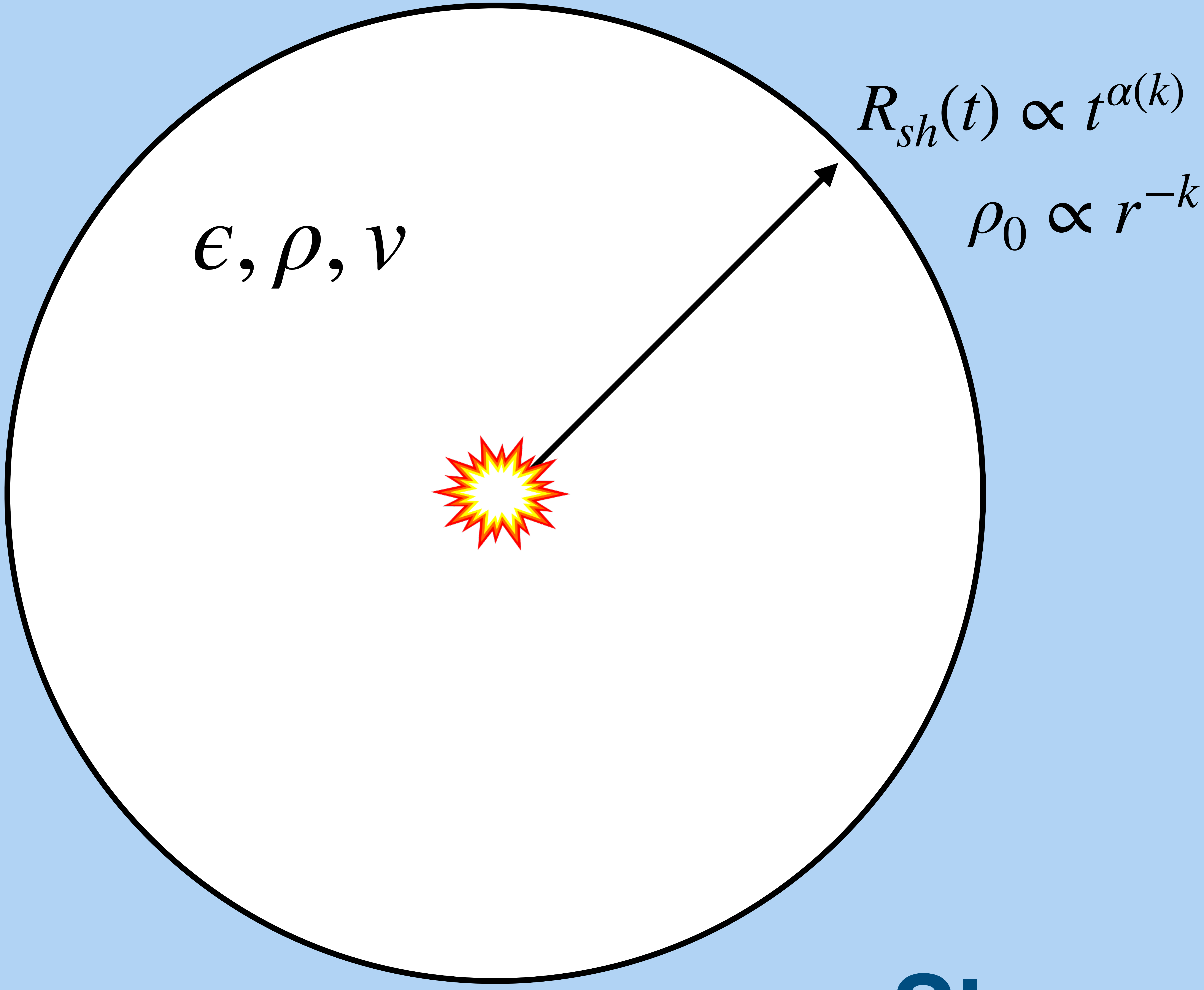
In the absence of `special' dimensional scales:

PDE → **variable transformation** → ODE

fluid equations
(r, t)

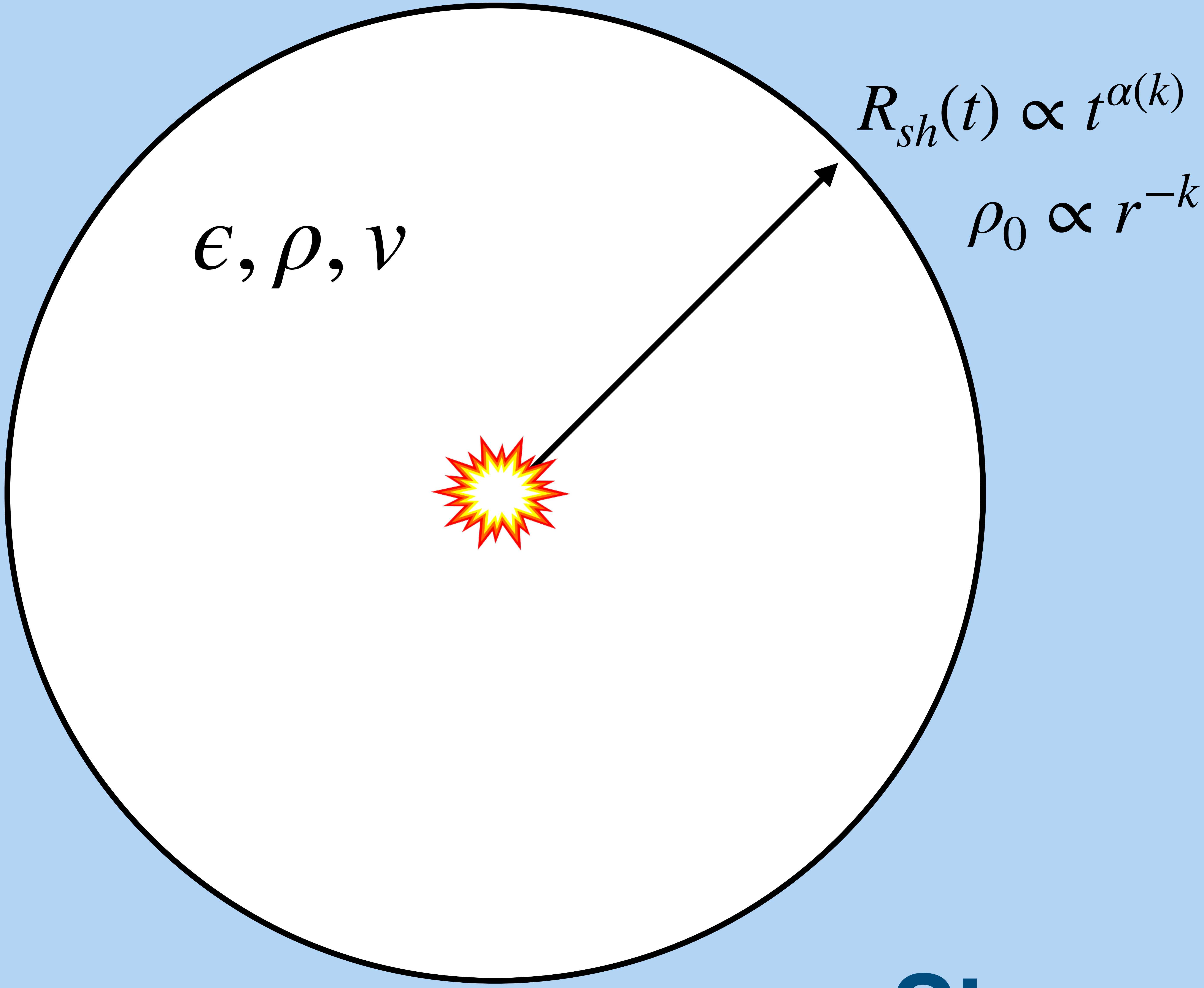
self-similar
ξ

Benefits: mathematical simplicity, sometimes analytical, simple asymptotic behaviour



(Newtonian)

Strong explosion problem



Similarity variable

Similarity exponent

$$\xi = \frac{r}{R_{sh}(t)}, \quad \alpha(k) = ?$$

(Newtonian)

Strong explosion problem

First & Second type solutions (Zel'dovich)

$$\rho_0 \propto r^{-k}, \quad R_{sh}(t) \propto t^{\alpha}$$

First & Second type solutions (Zel'dovich)

First type

Dimensional analysis

Energy conservation

$$\int_0^{R_{sh}} \epsilon \, dV \propto t^{\alpha(5-k)-2} I(\xi) = constant$$

$$\alpha = \frac{2}{5-k}$$

Sedov-von Neumann-Taylor

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Second type

Eigenvalue problem

Shock loses causal contact with blast wave

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Eliminate singularity in ODEs

(α numerical)

Waxman & Shvartz 1993

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Solution gap: 3<k<3.25

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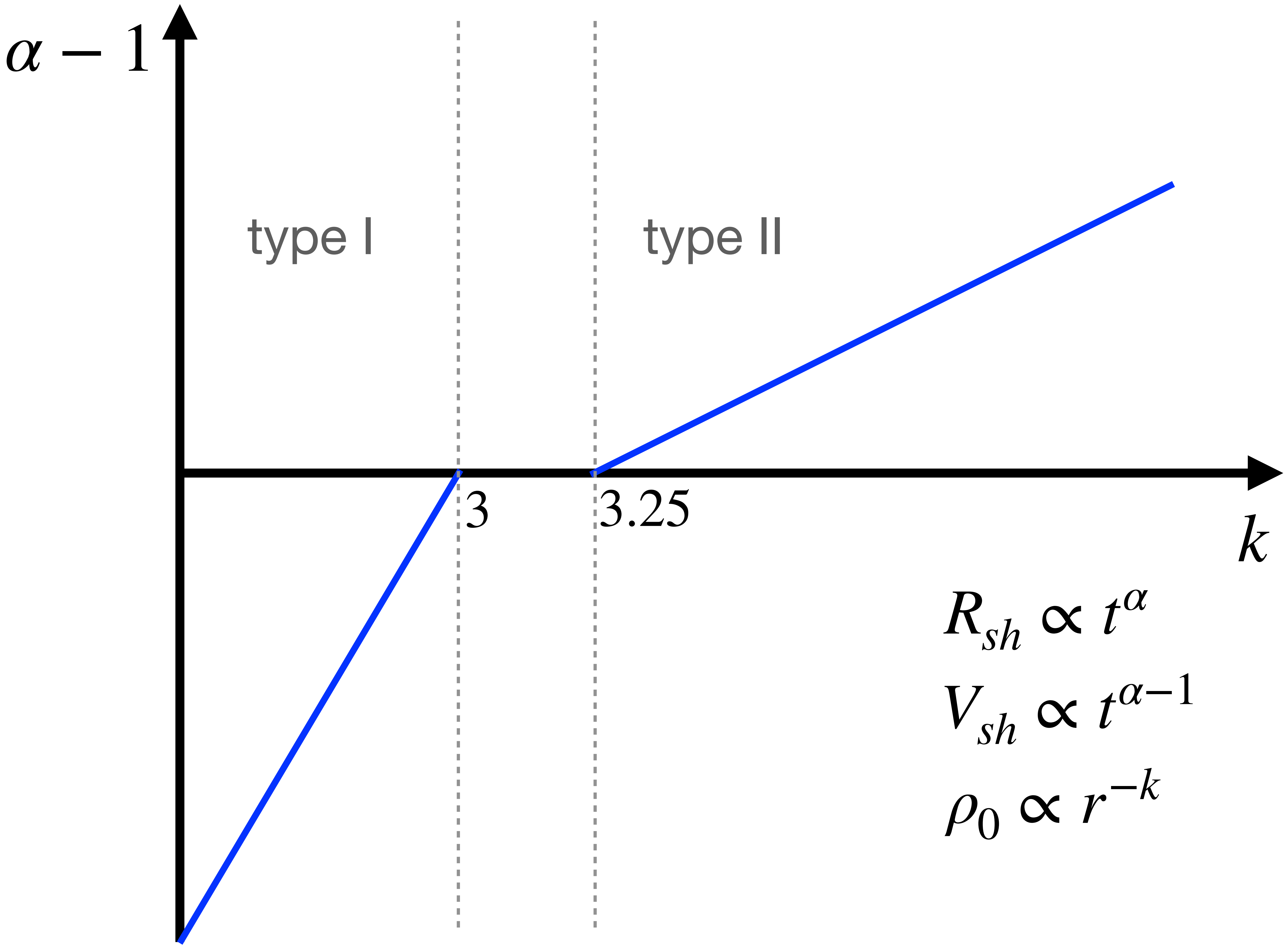
Third type solutions: 3<k<3.25 (Gruzinov 2003)



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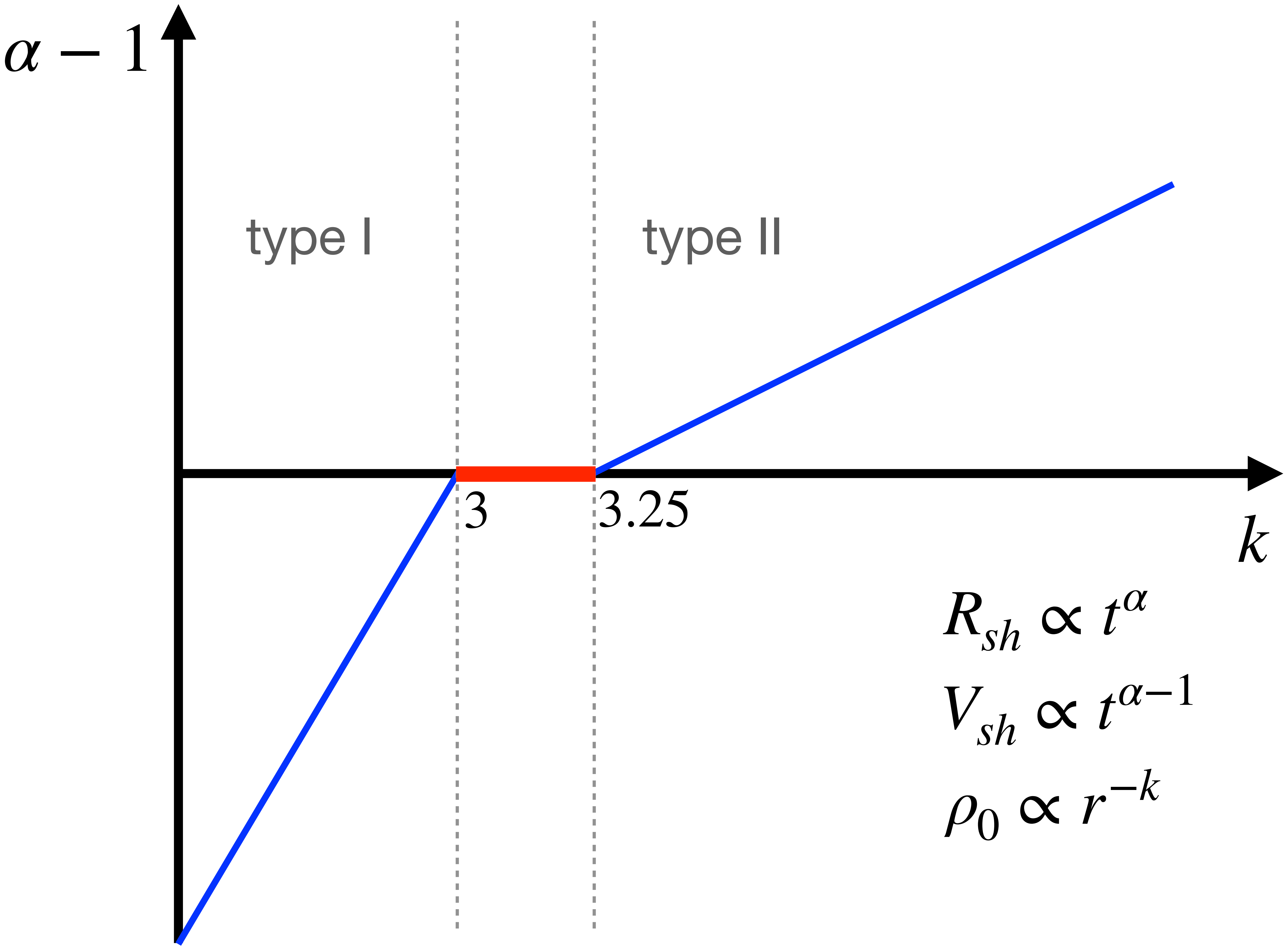

$$R_{sh} \propto t^{\alpha}$$

$$V_{sh} \propto t^{\alpha-1}$$

$$\rho_0 \propto r^{-k}$$

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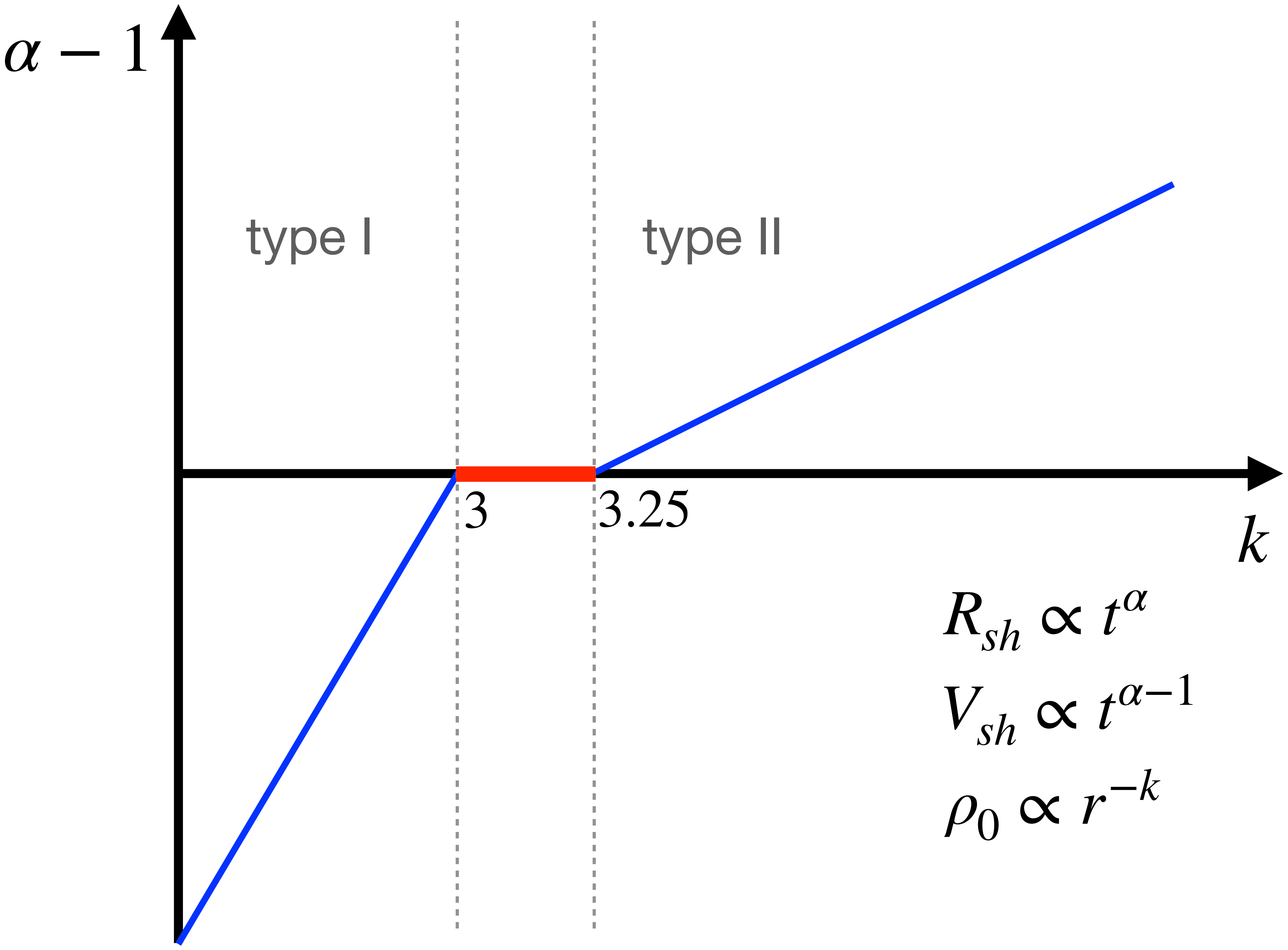

an infinitely heavy **piston**

(lies outside self-similar domain)

$$v = const \rightarrow \alpha - 1 = 0$$

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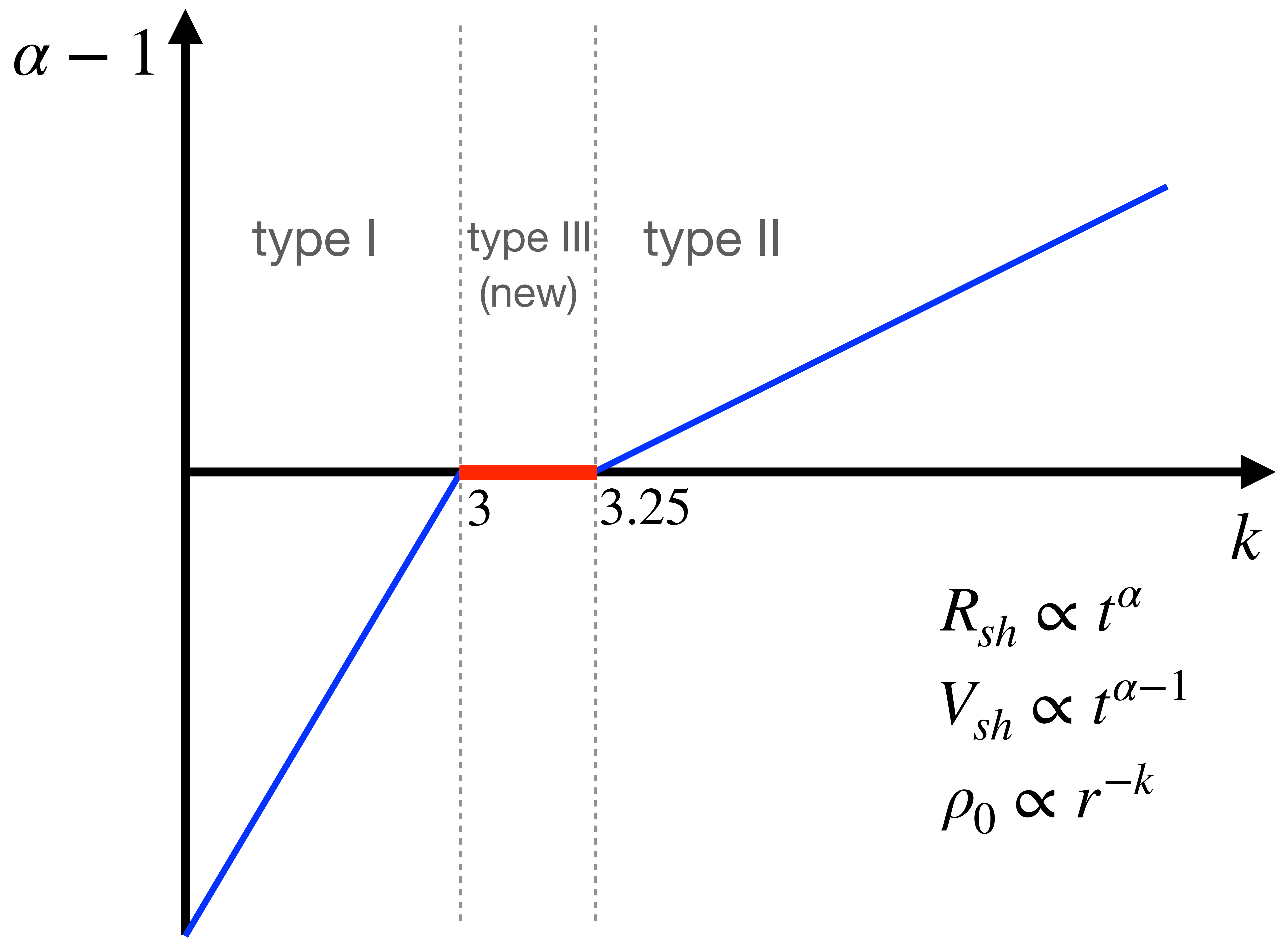

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Type III: self-similar exponents determined by the non self-similar part of the flow

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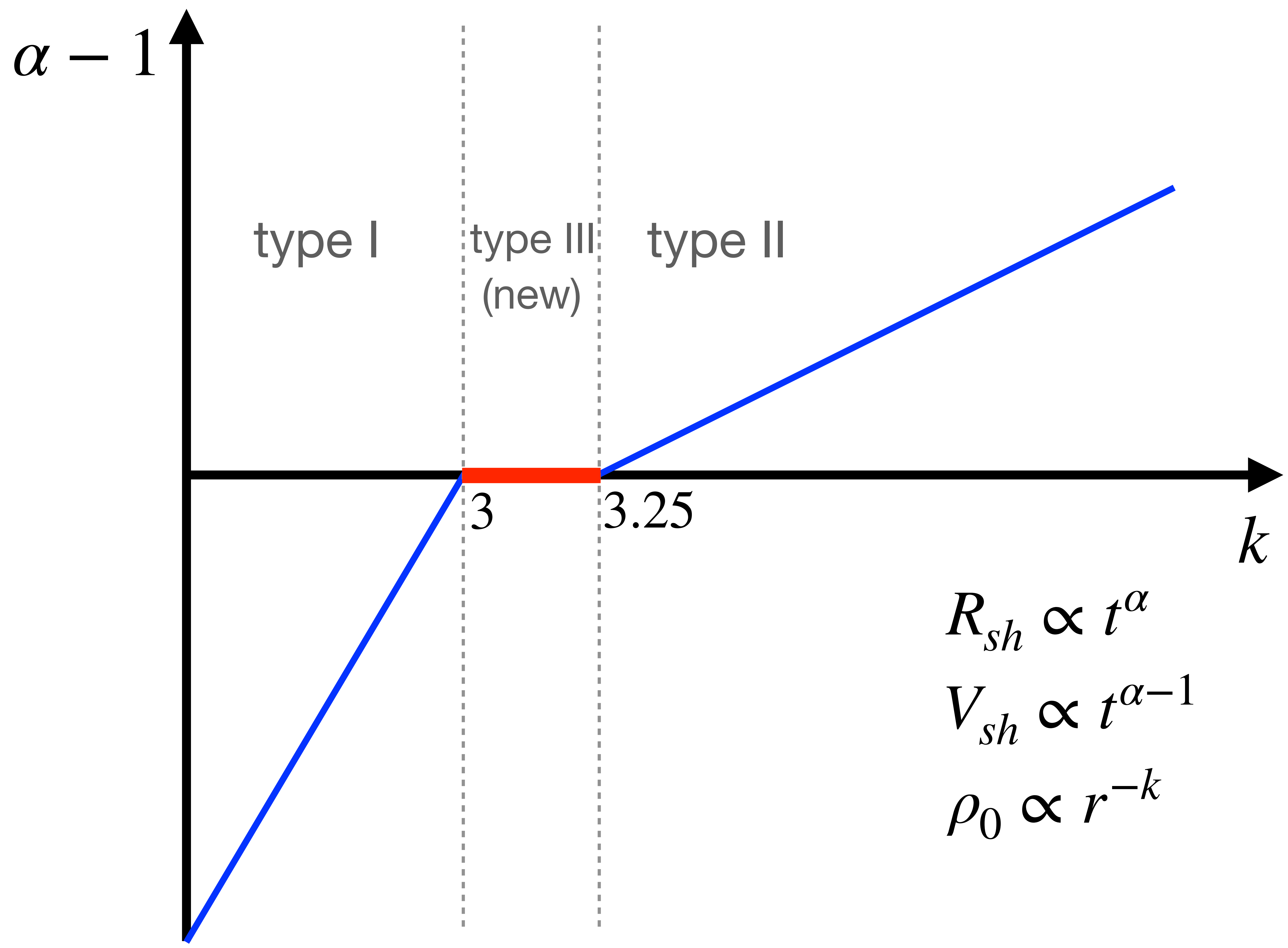

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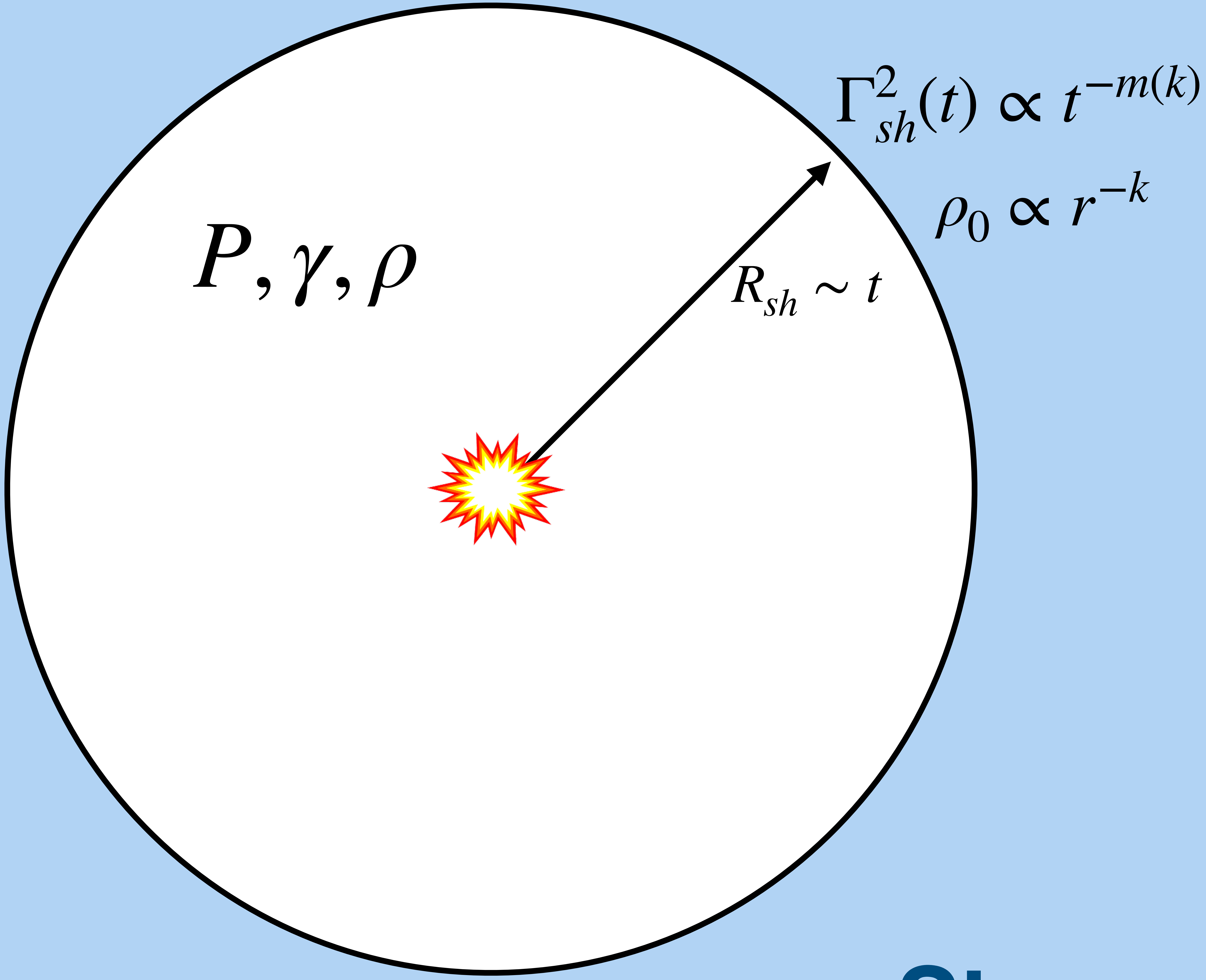
Trivial type III

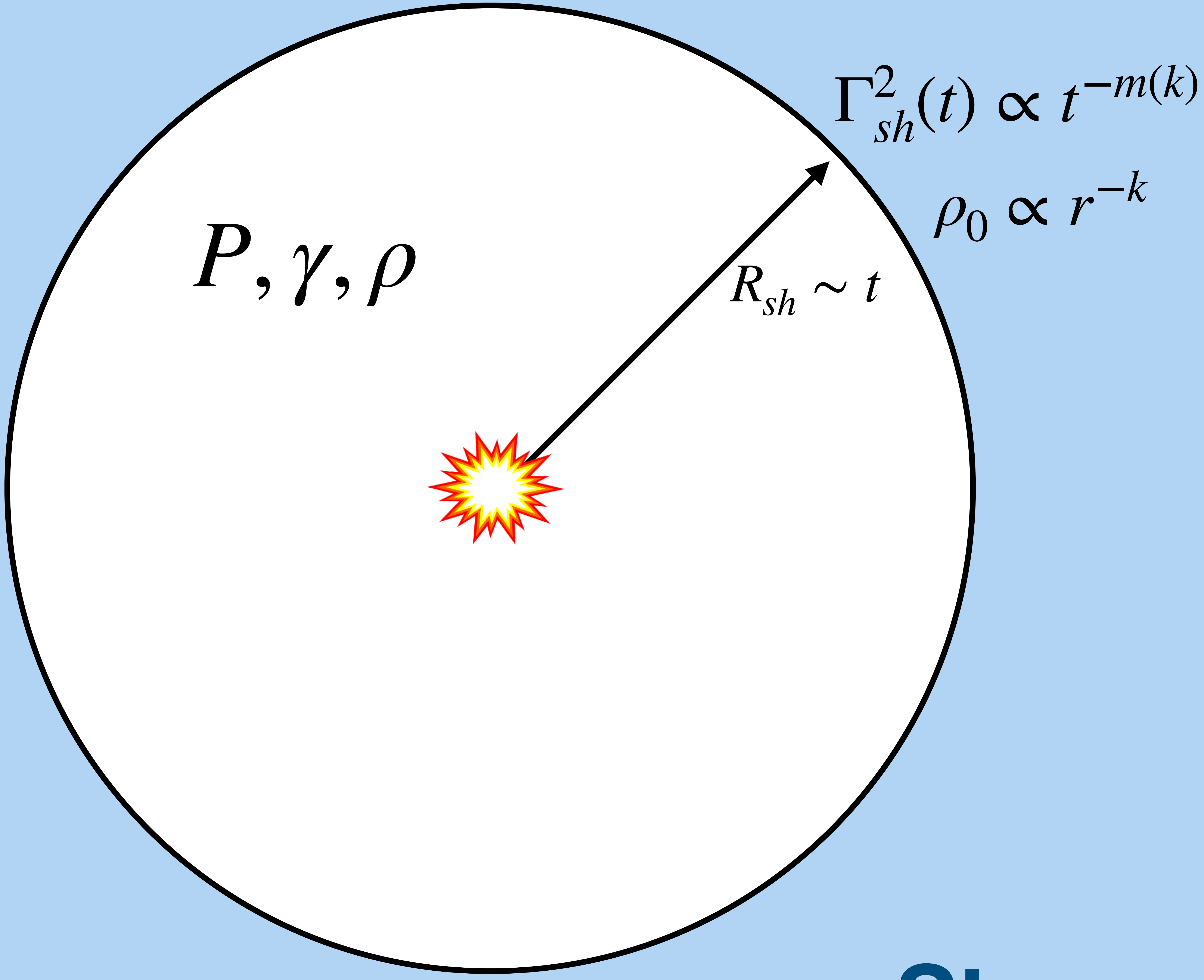
Type III: self-similar exponents determined by the non self-similar part of the flow

Ultra-relativistic blast waves



(UR)

Strong explosion problem



Similarity variable

Similarity exponent

$$\chi = \frac{\Delta r_{sh}}{R_{sh}/\Gamma^2_{sh}}, \quad m(k) = ?$$

(UR)

Strong explosion problem

(In plane-parallel geometry)

$$P, \gamma, \rho$$

$$R_{sh} \sim t$$

$$\Gamma^2_{sh}(t) \propto t^{-m(k)}$$

$$\rho_0 \propto r^{-k}$$

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Strong explosion problem

First type solutions (Blandfold & McKee 1976)

$$E \sim \int_0^{R_{sh}} 4\gamma^2 P dr \propto t^{1-m-k} \left. \chi^{\frac{4k-7}{3(2-k)}} \right|_1^{\infty}$$

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eliminate singularity

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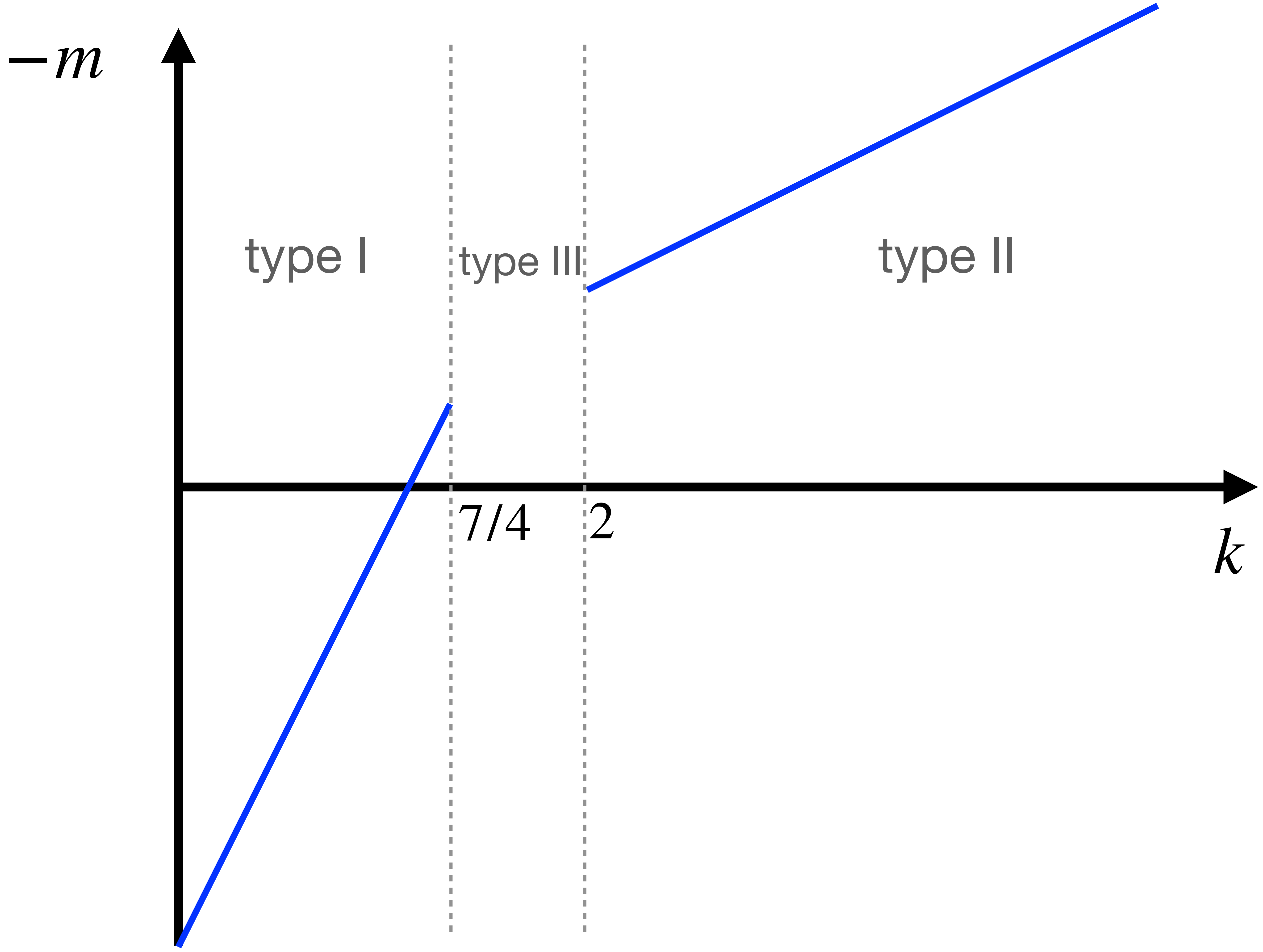

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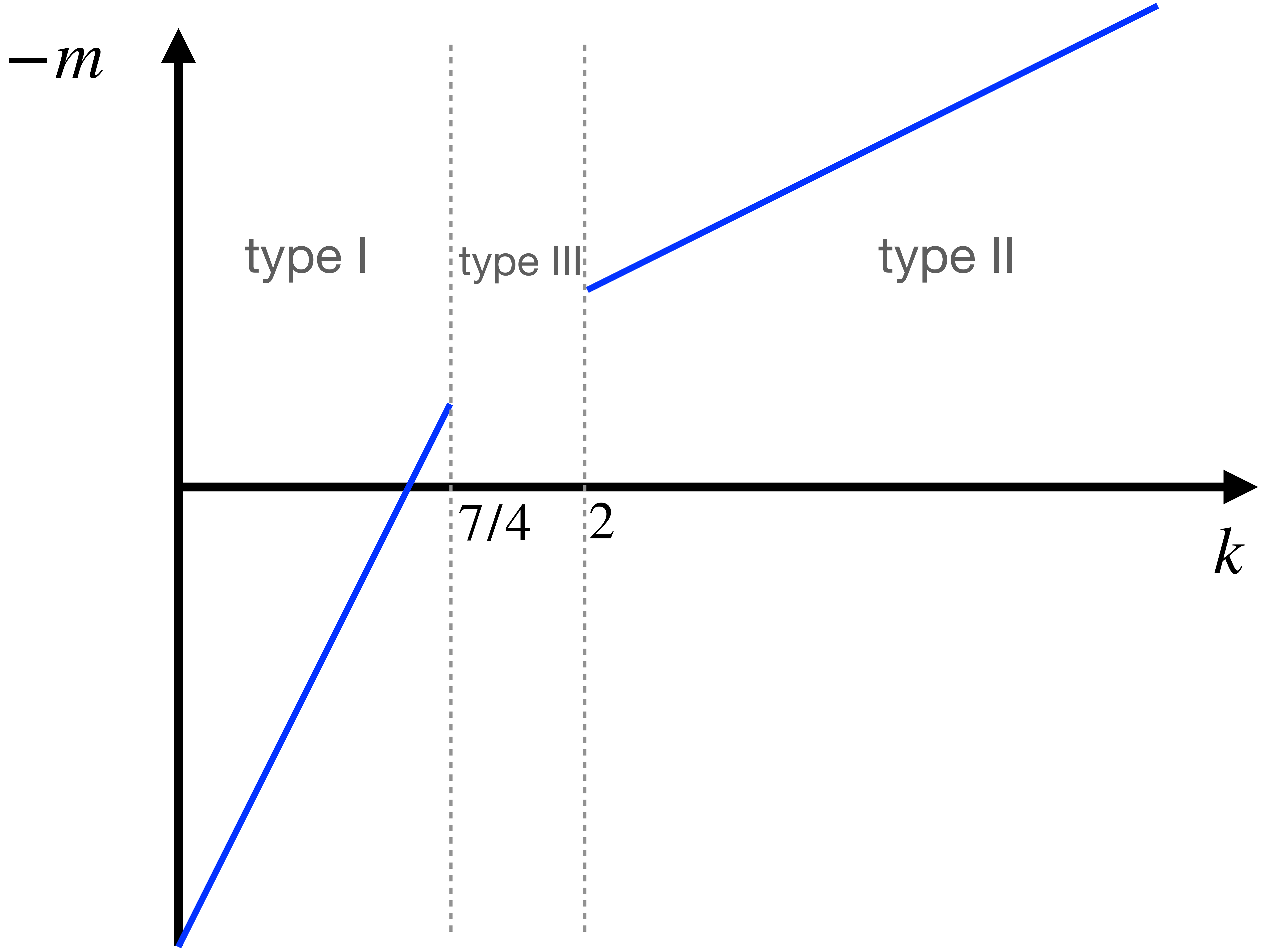

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non-trivial type III

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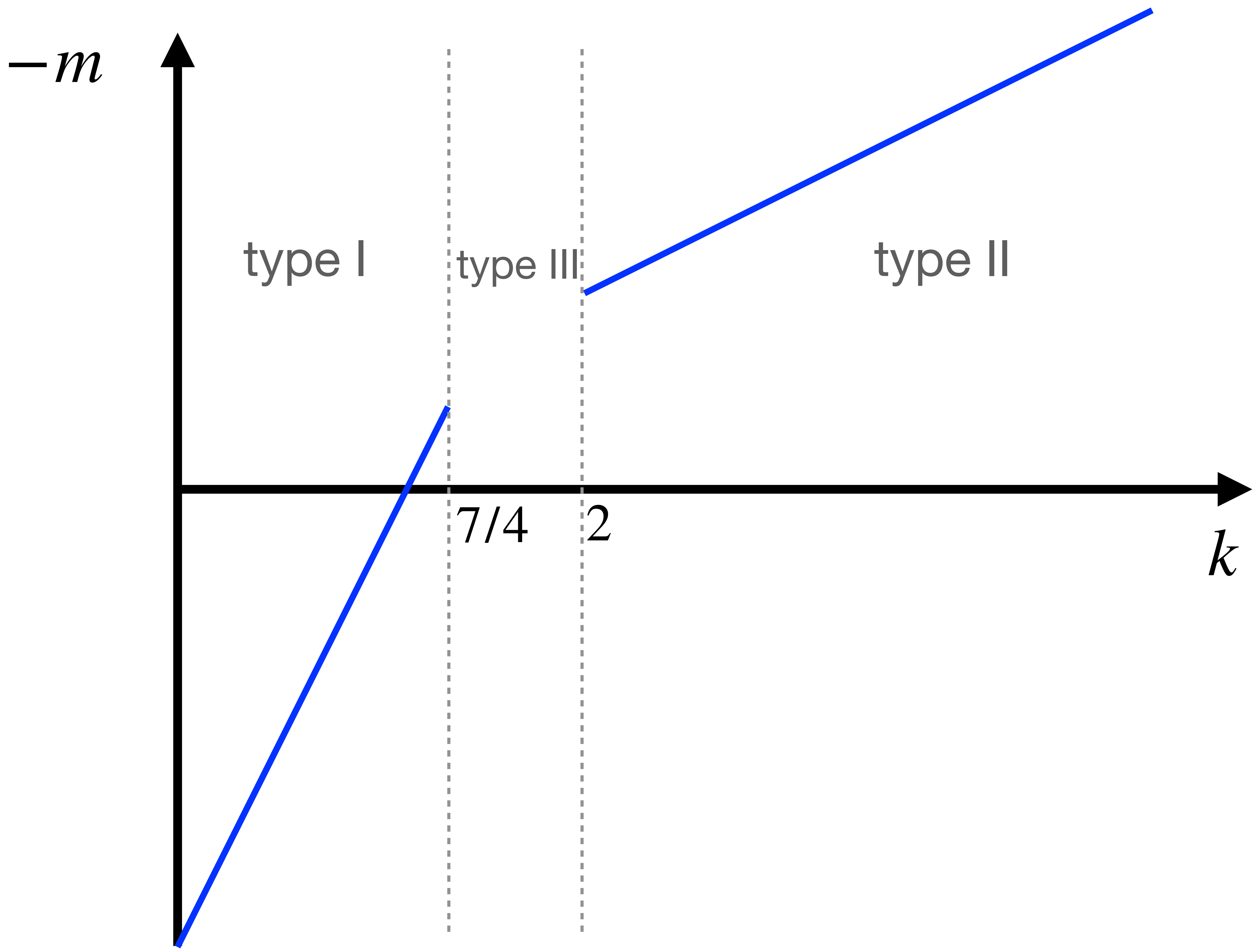

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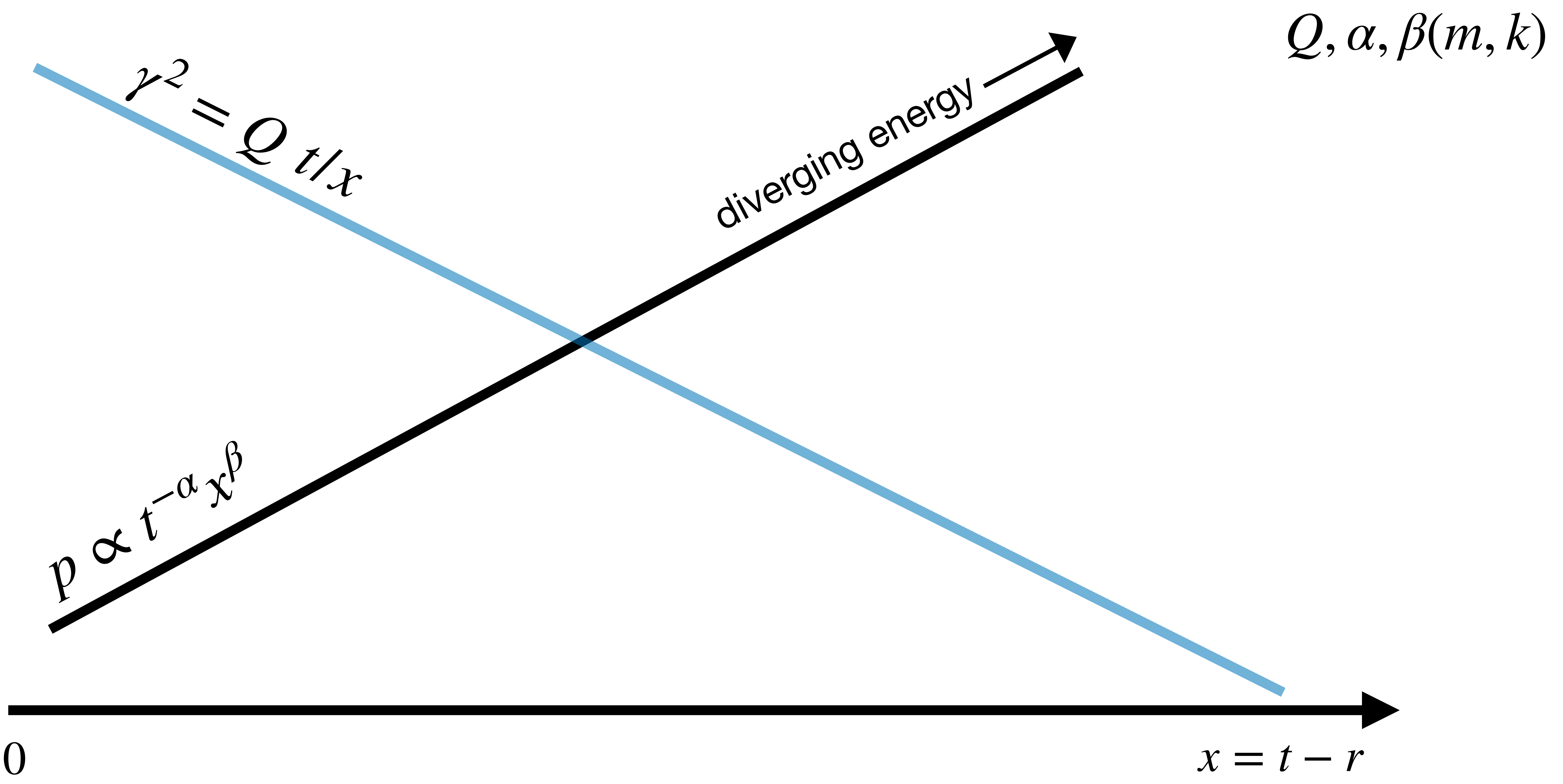
non-trivial type III

Hot & accelerating UR gas

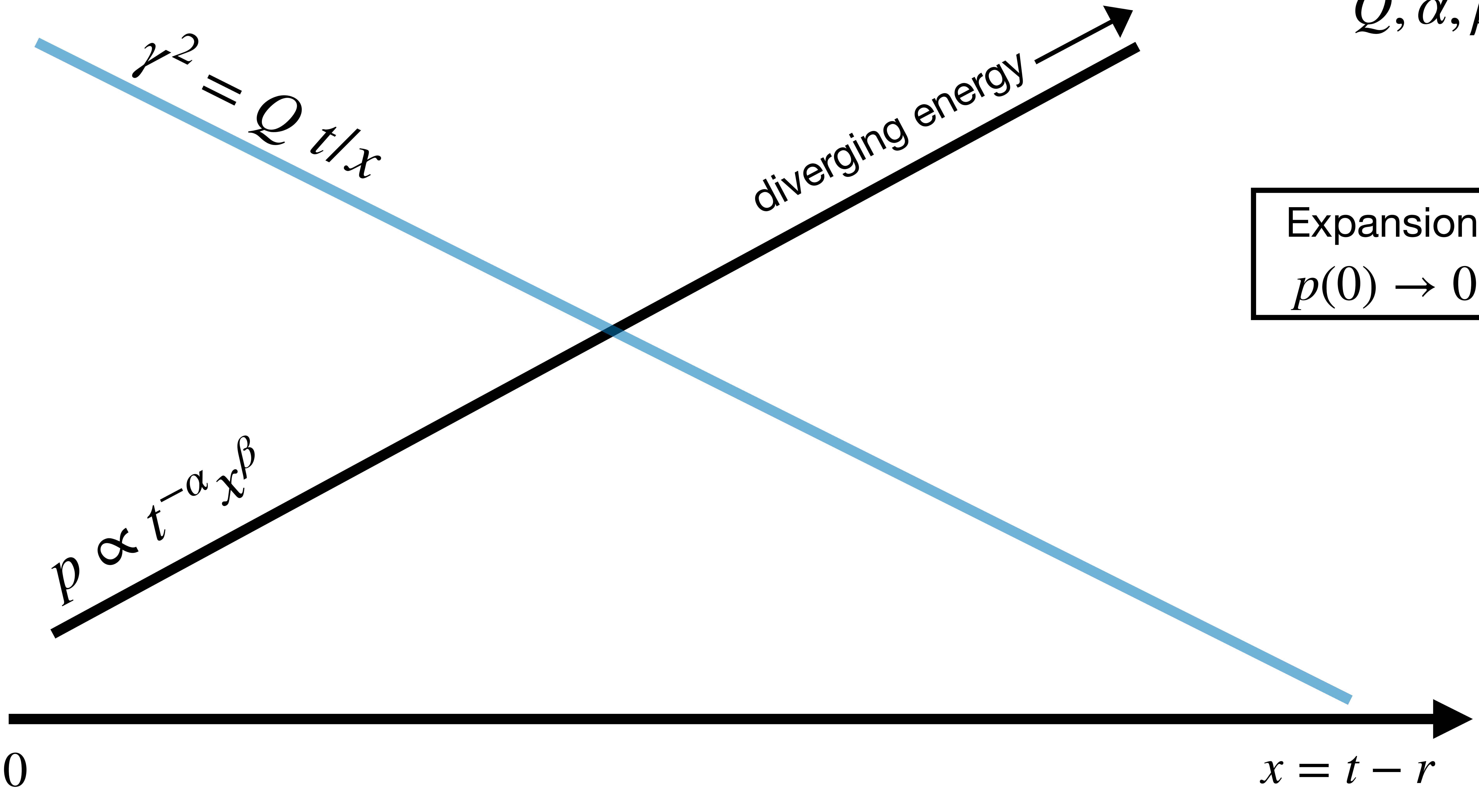
Thermal ⟷ Bulk

Asymptotic solution — far behind the shock



Asymptotic solution — far behind the shock

$Q, \alpha, \beta(m, k)$



Expansion into vacuum
$p(0) \to 0\ , \gamma^2(0) \to \infty$

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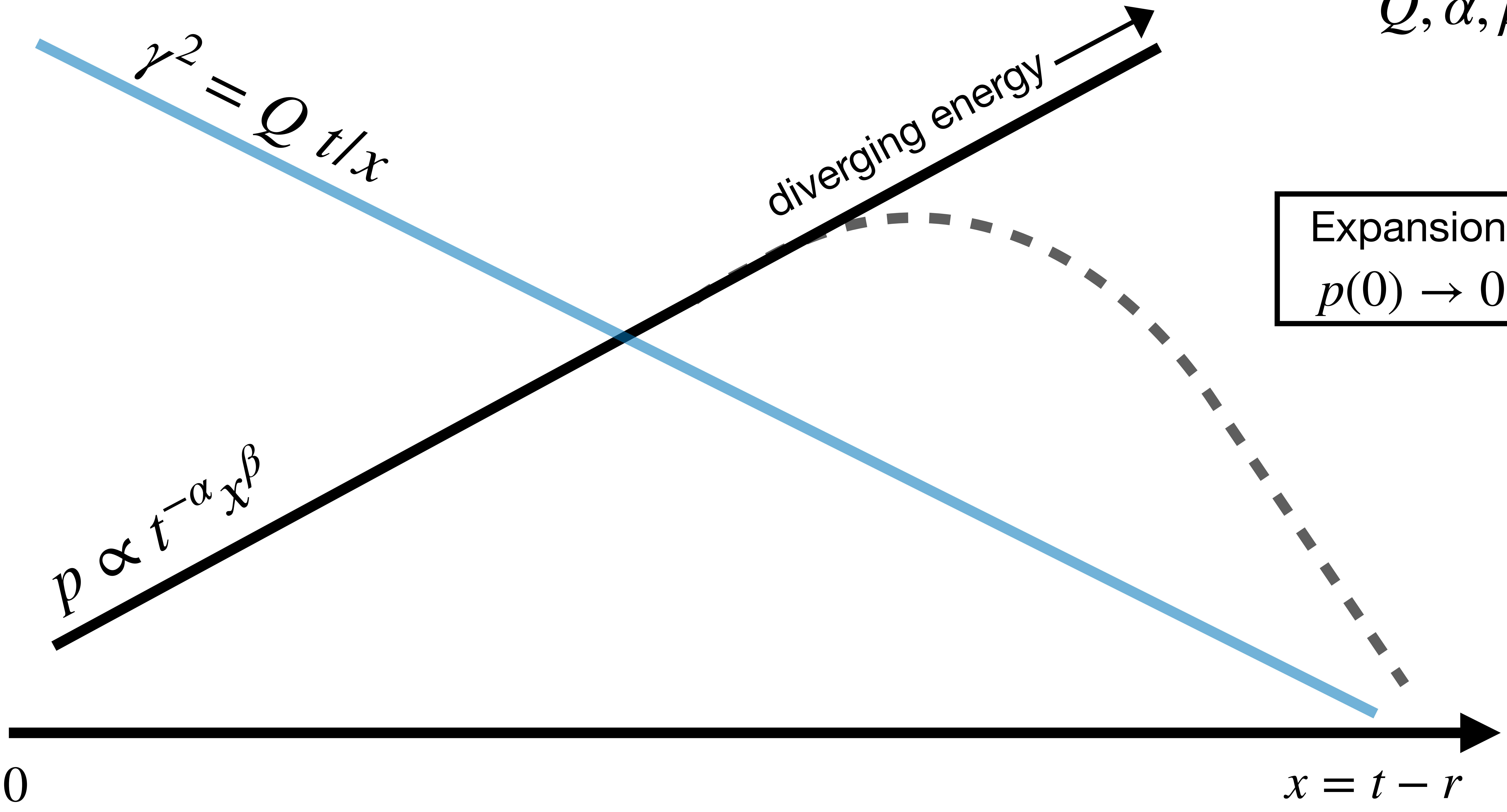

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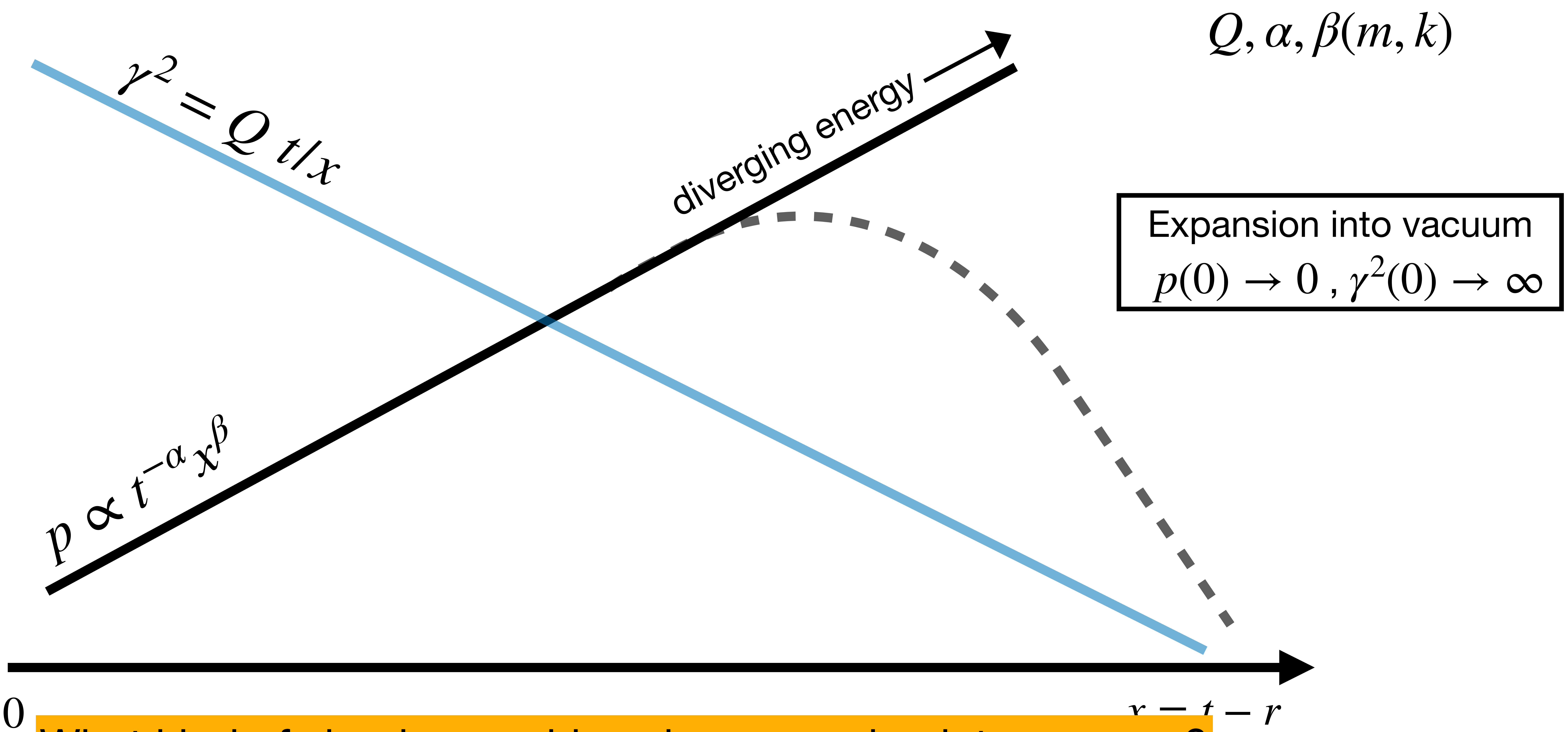

What kind of shocks are driven by expansion into vacuum?

Exact solution (inspired by Landau & Lifshitz)

UR fluid eqns

$$\begin{cases} 4\dfrac{\partial \gamma^2 p}{\partial t} + \dfrac{\partial p}{\partial x} = 0 \\ \dfrac{\partial p}{\partial t} + \dfrac{\partial p/\gamma^2}{\partial x} = 0 \end{cases}$$

$x = t - r$

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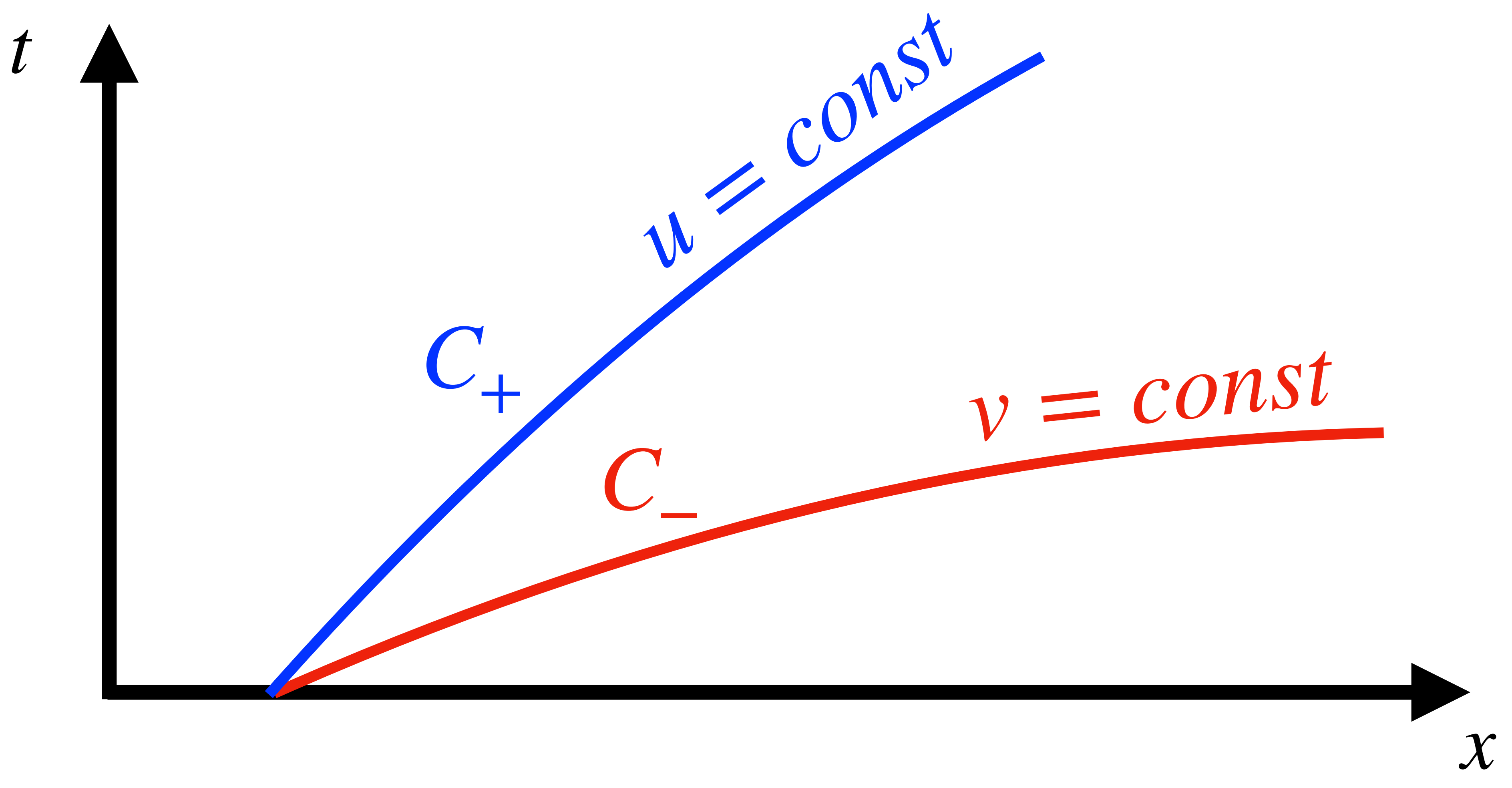
Klein-Gordon eqn

$$\boxed{\partial_{uv}\psi = \psi}$$

$\{u, v\}$: Riemann invariants



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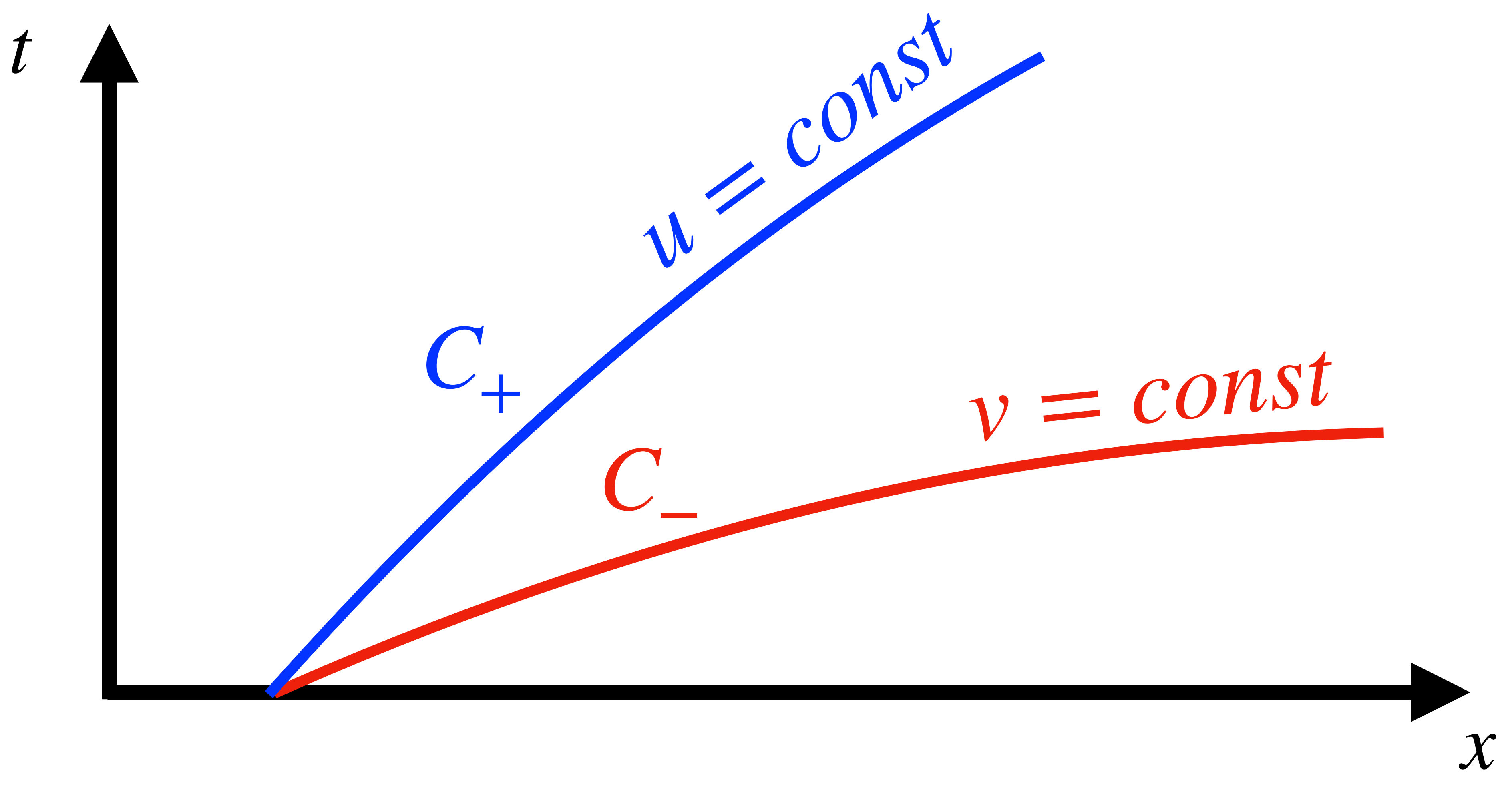

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Applying boundary conditions on $C_\pm$:

$$\psi = -\sum_{n=1}^{\infty} \frac{(v/\lambda)^n}{n!} \sum_{k=0}^{n-1} \frac{(u\lambda)^k}{k!}$$

Exact solution describing expansion into vacuum

Expansion into vacuum

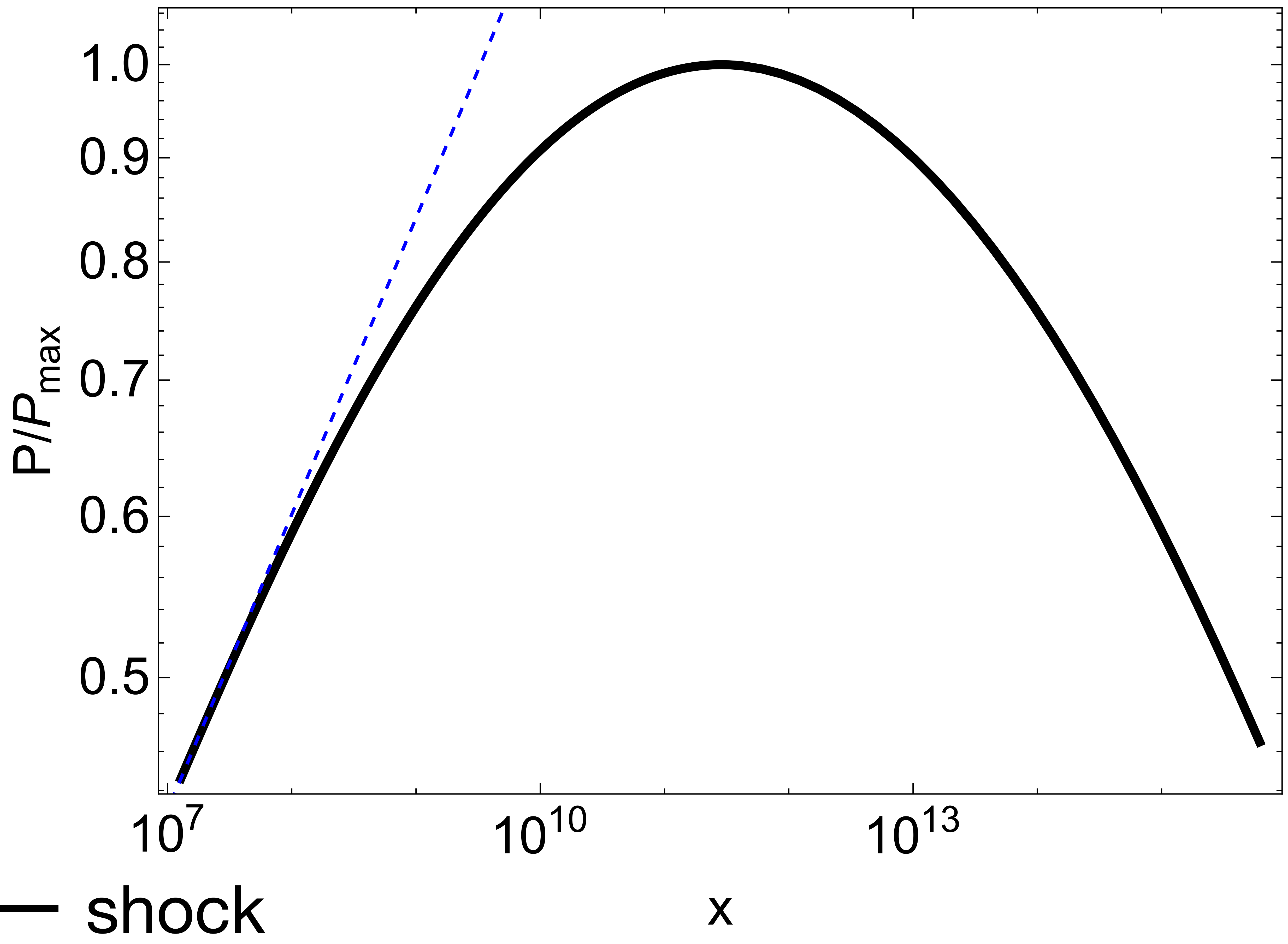
$$\psi \propto \begin{cases} e^{\lambda u + v/\lambda}, \ v \gg u\lambda^2 \\ e^{2\sqrt{uv}}, \ v \ll u\lambda^2 \end{cases}$$

powerlaw asymptotic

profile containing p_{max}

$\lambda(m, k)$

Finite energy ✔



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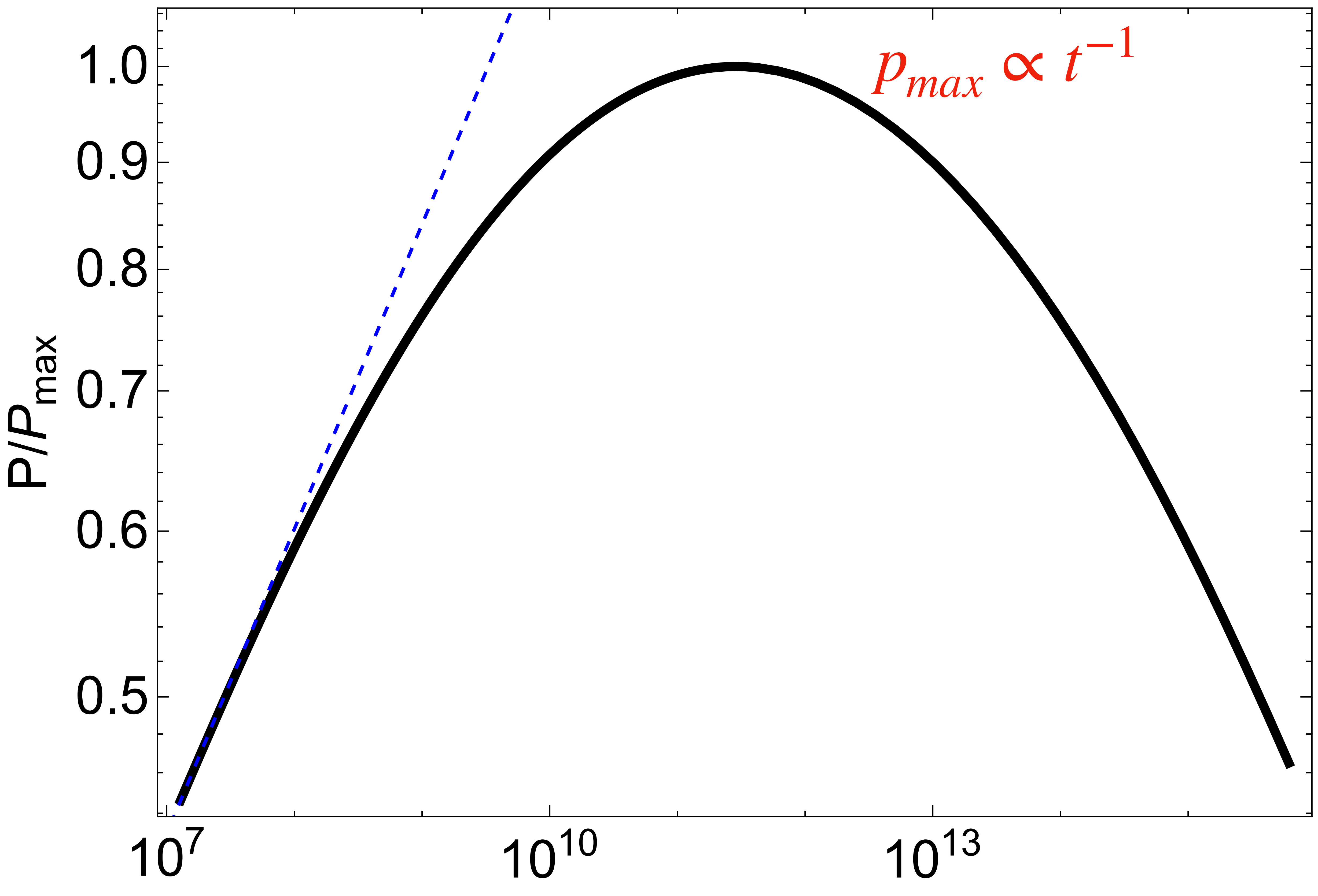
Finite energy ✔

new universal exponents:



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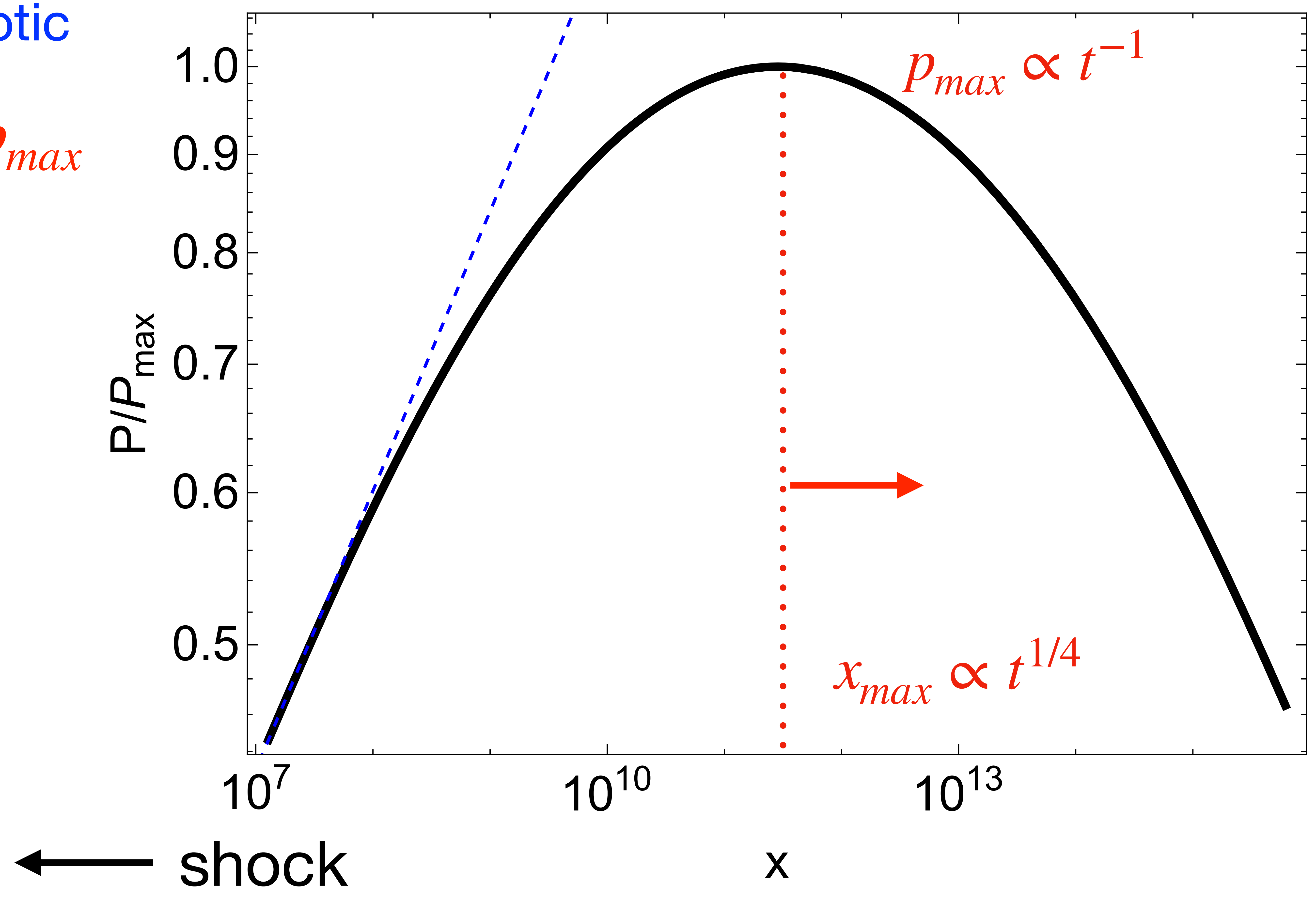

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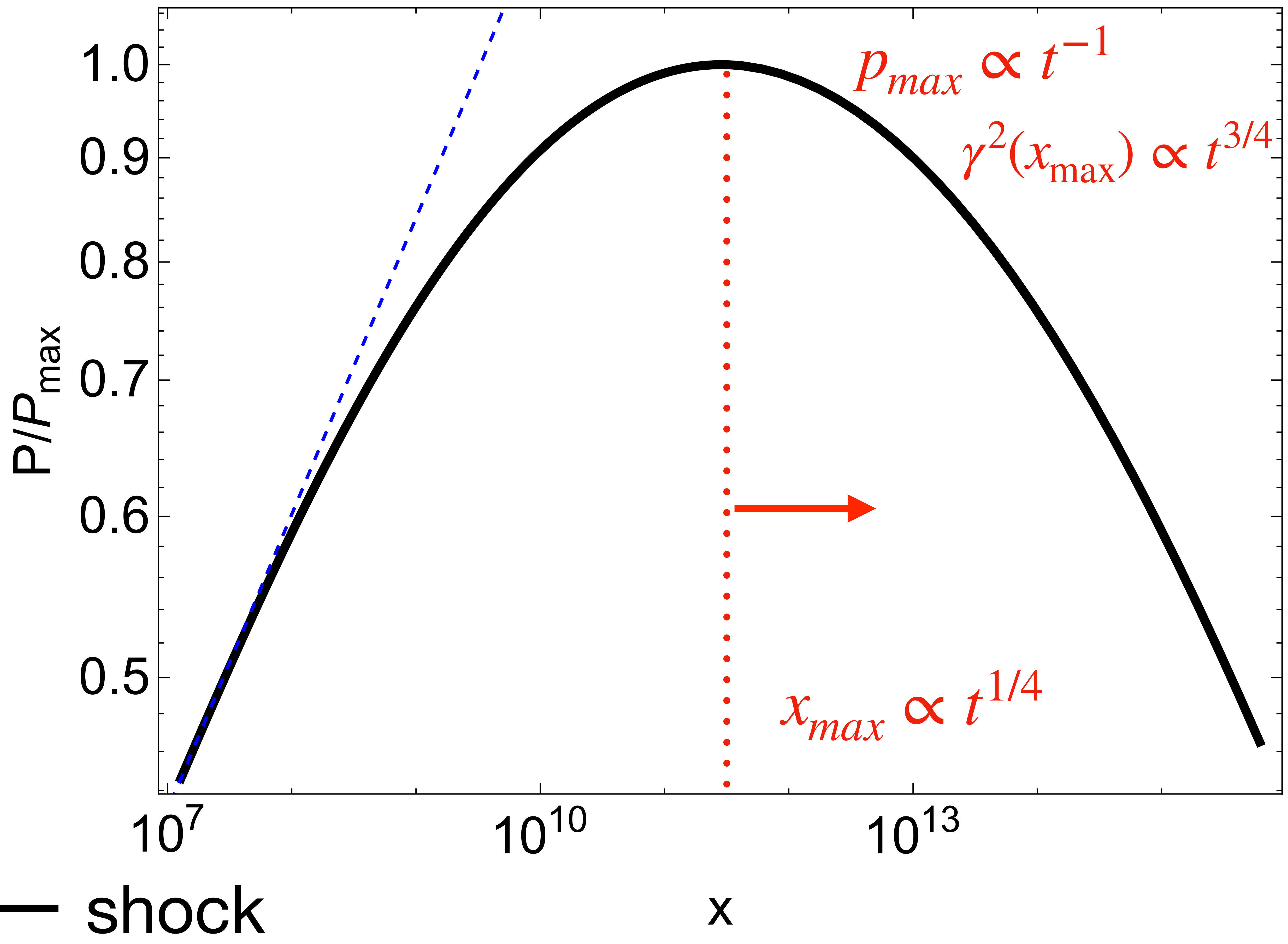
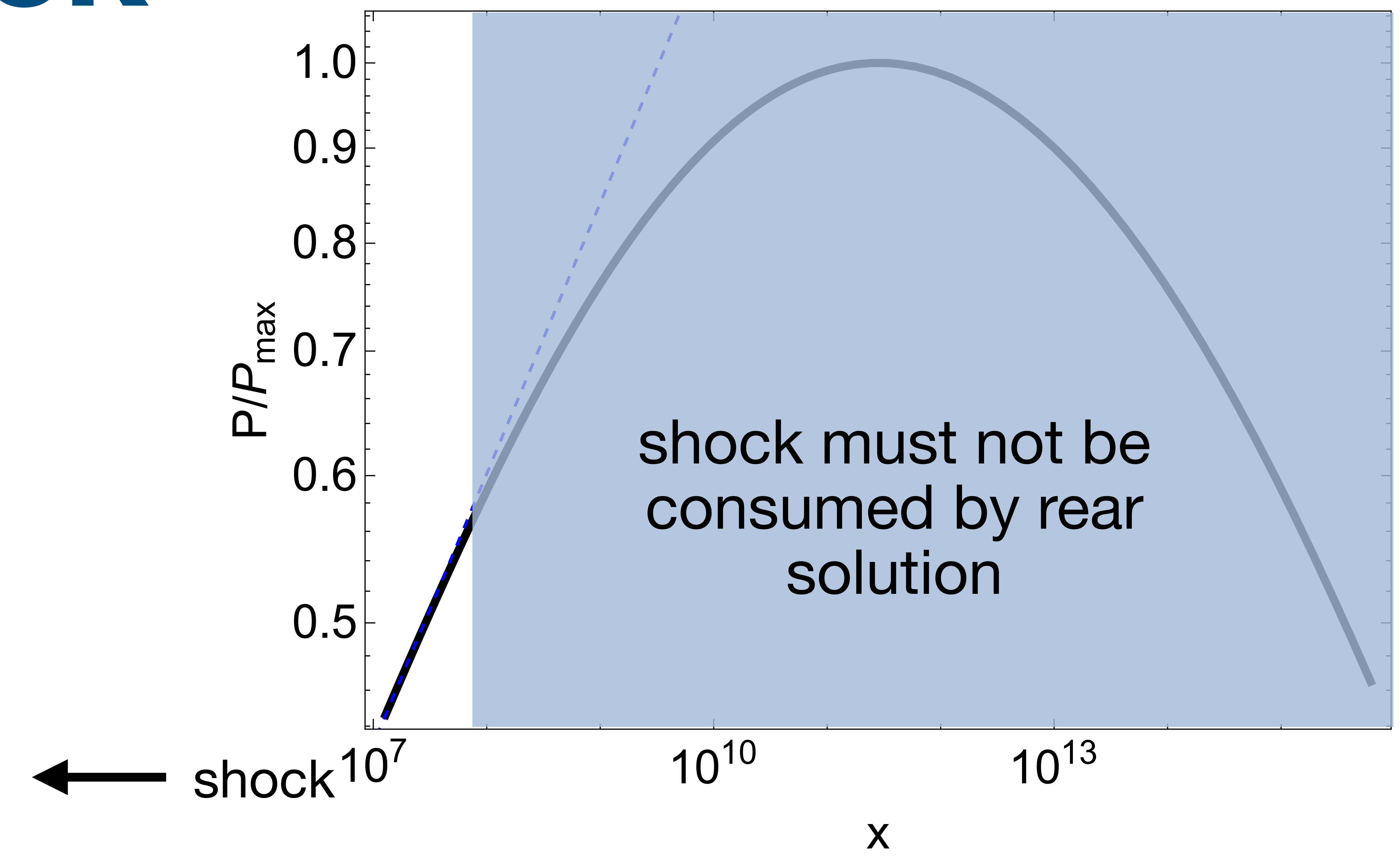


← shock

Consistency with a shock

What is the condition on $m(k)$?



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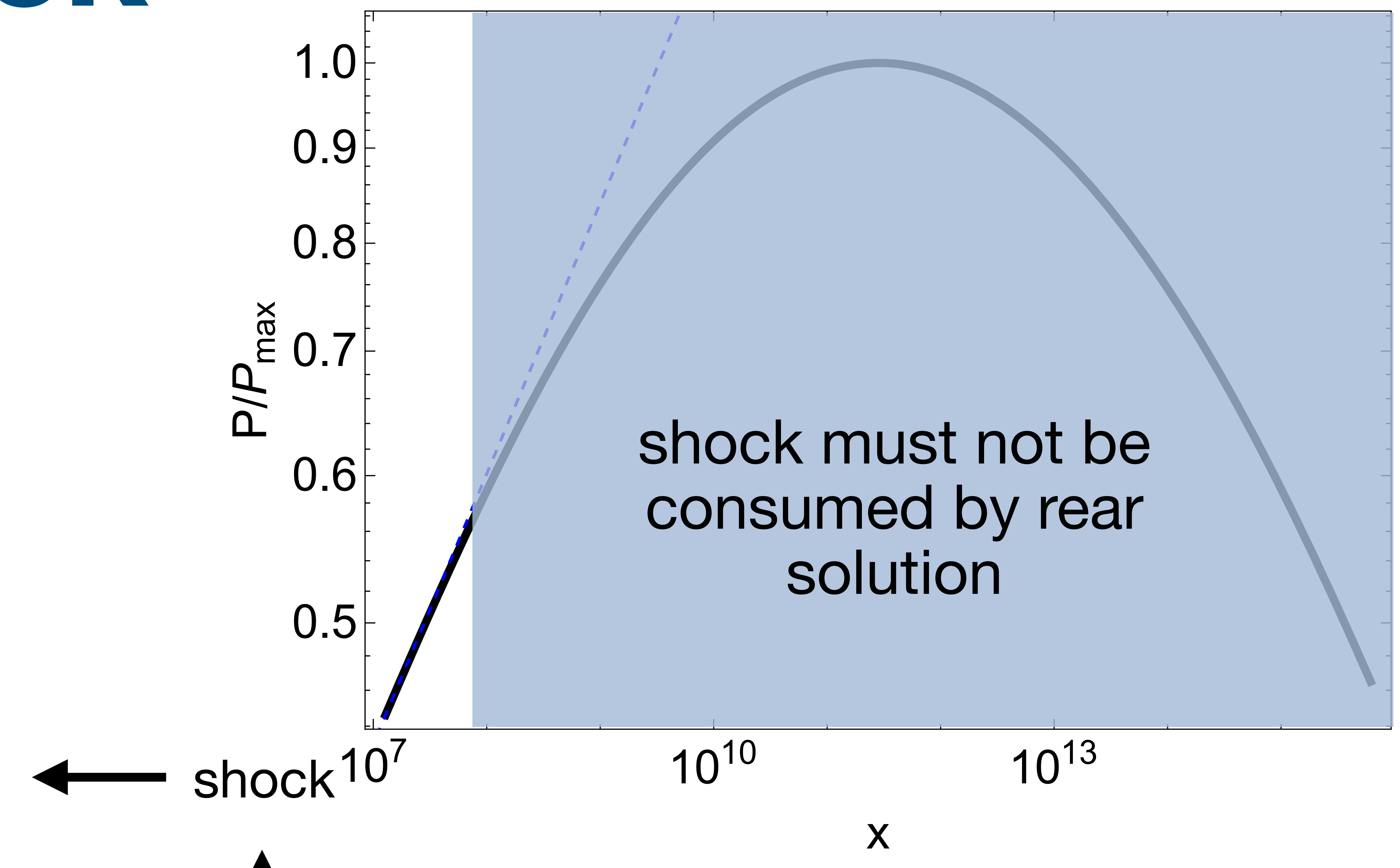

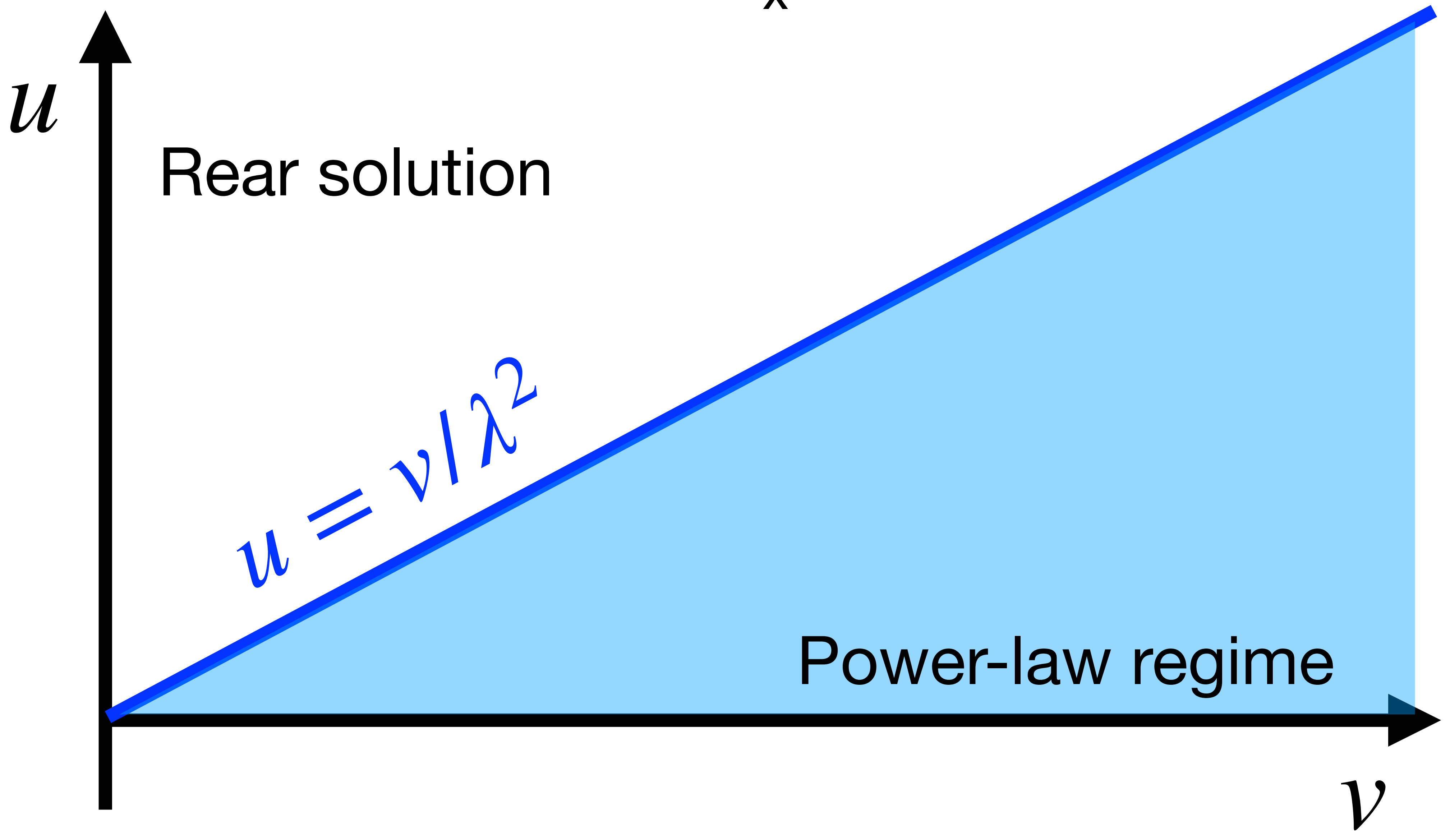

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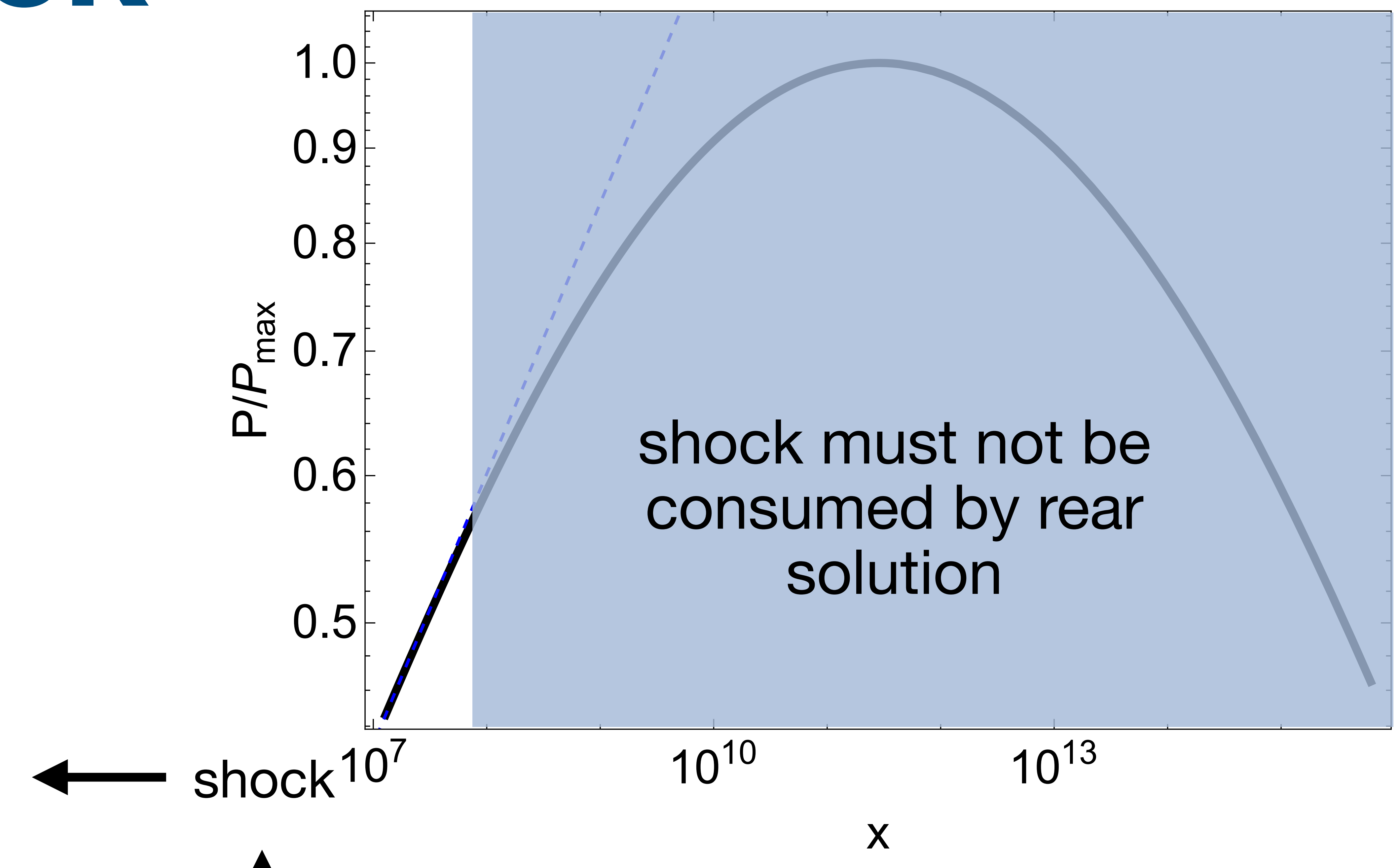

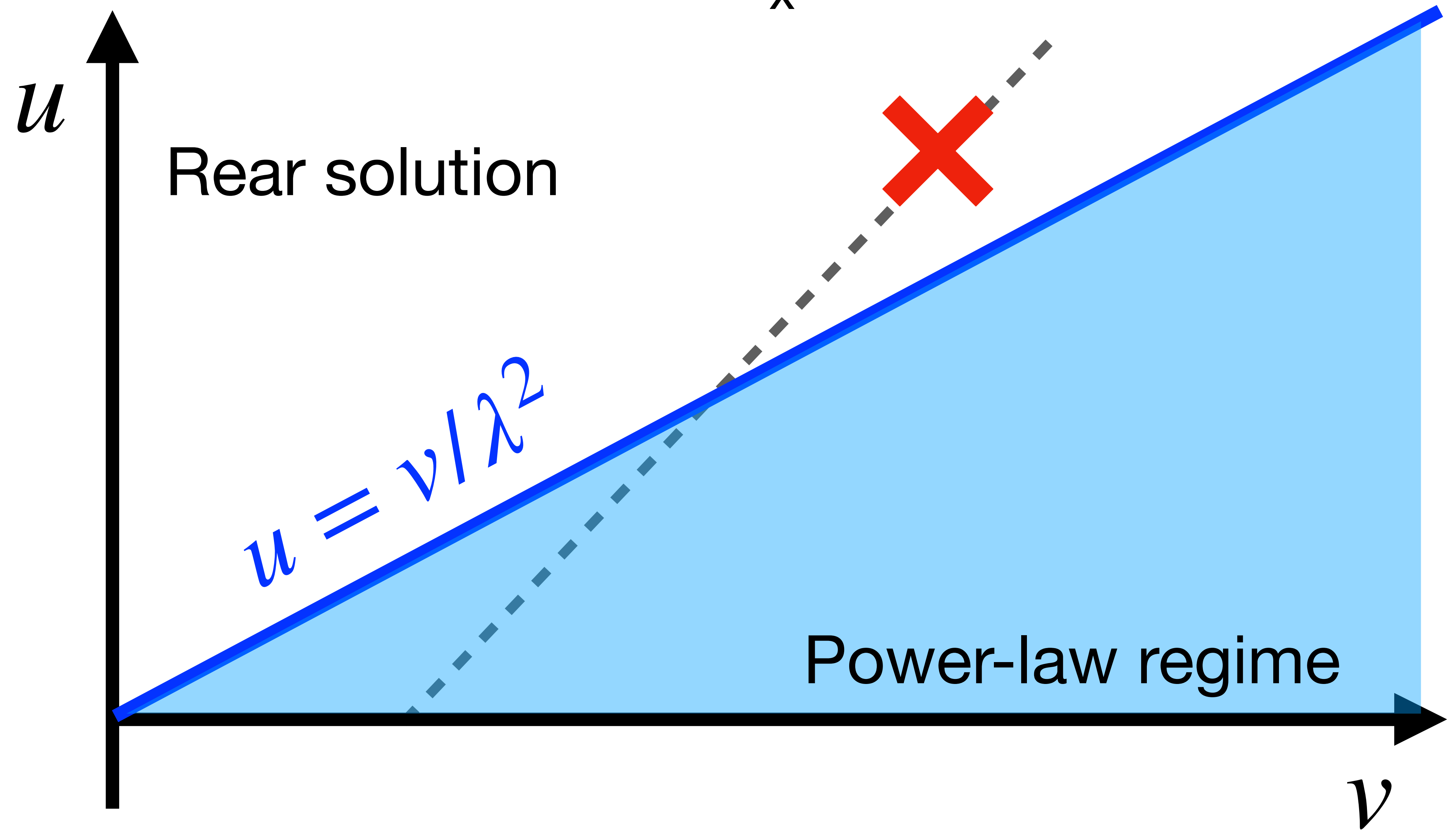

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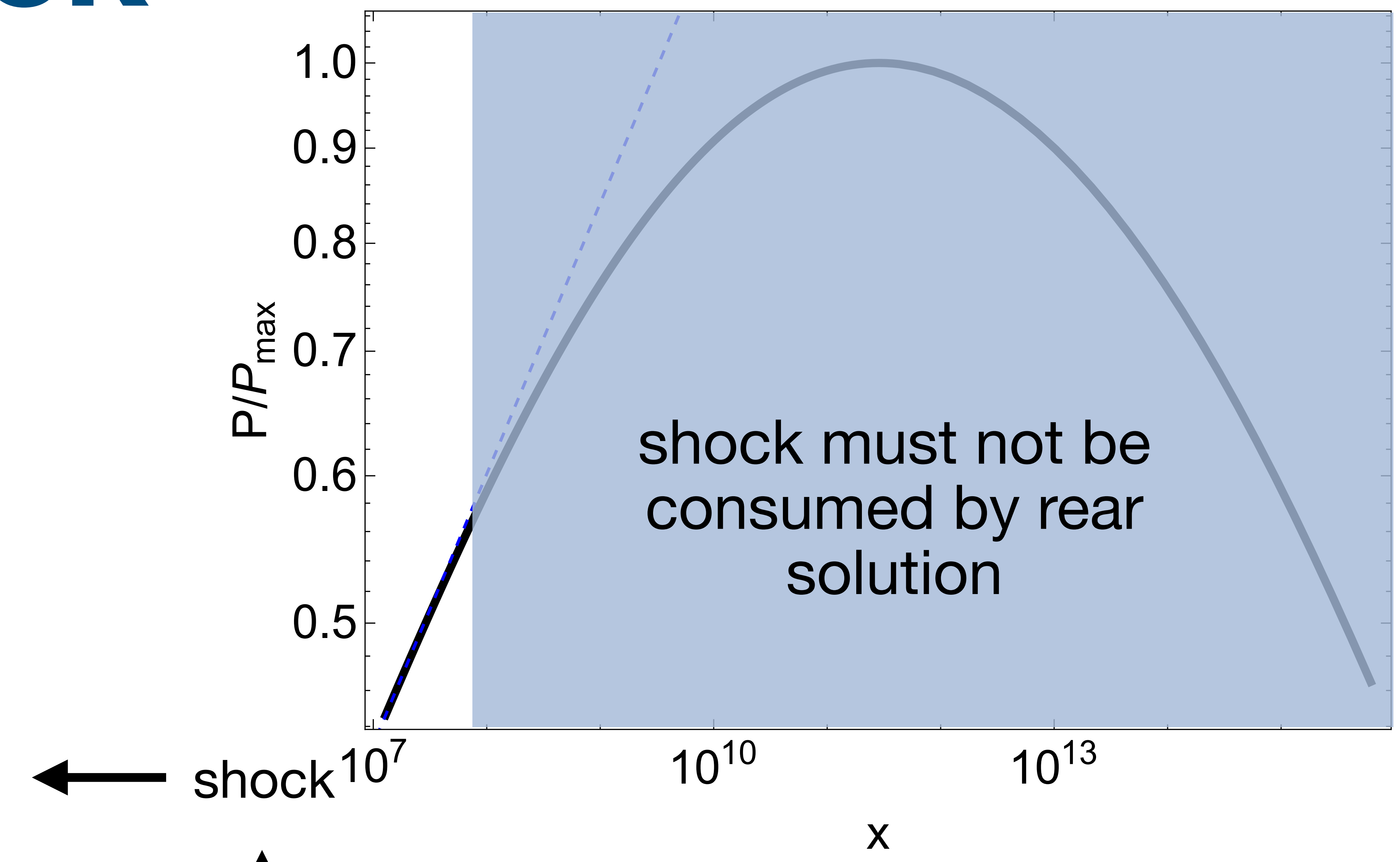

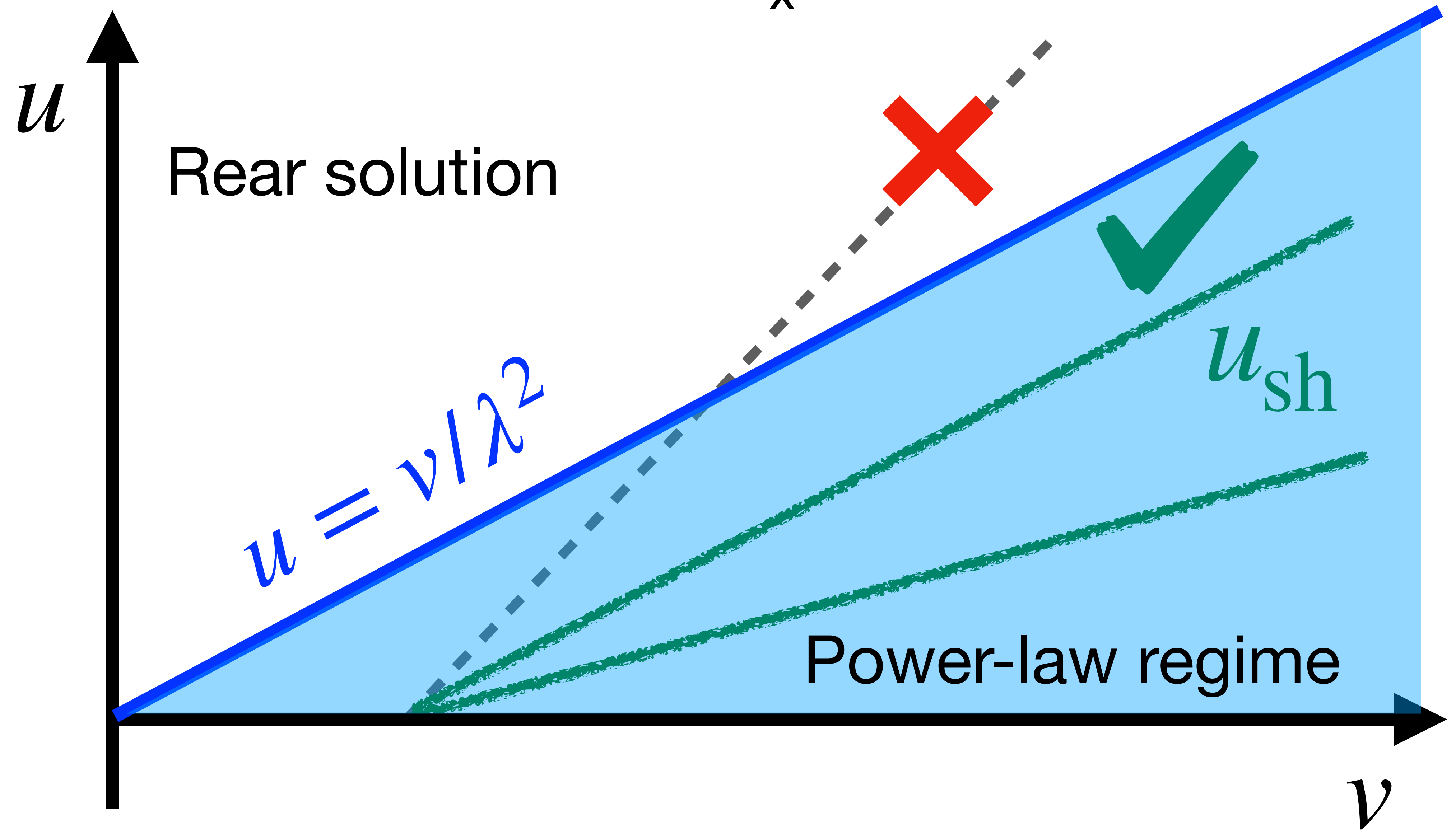

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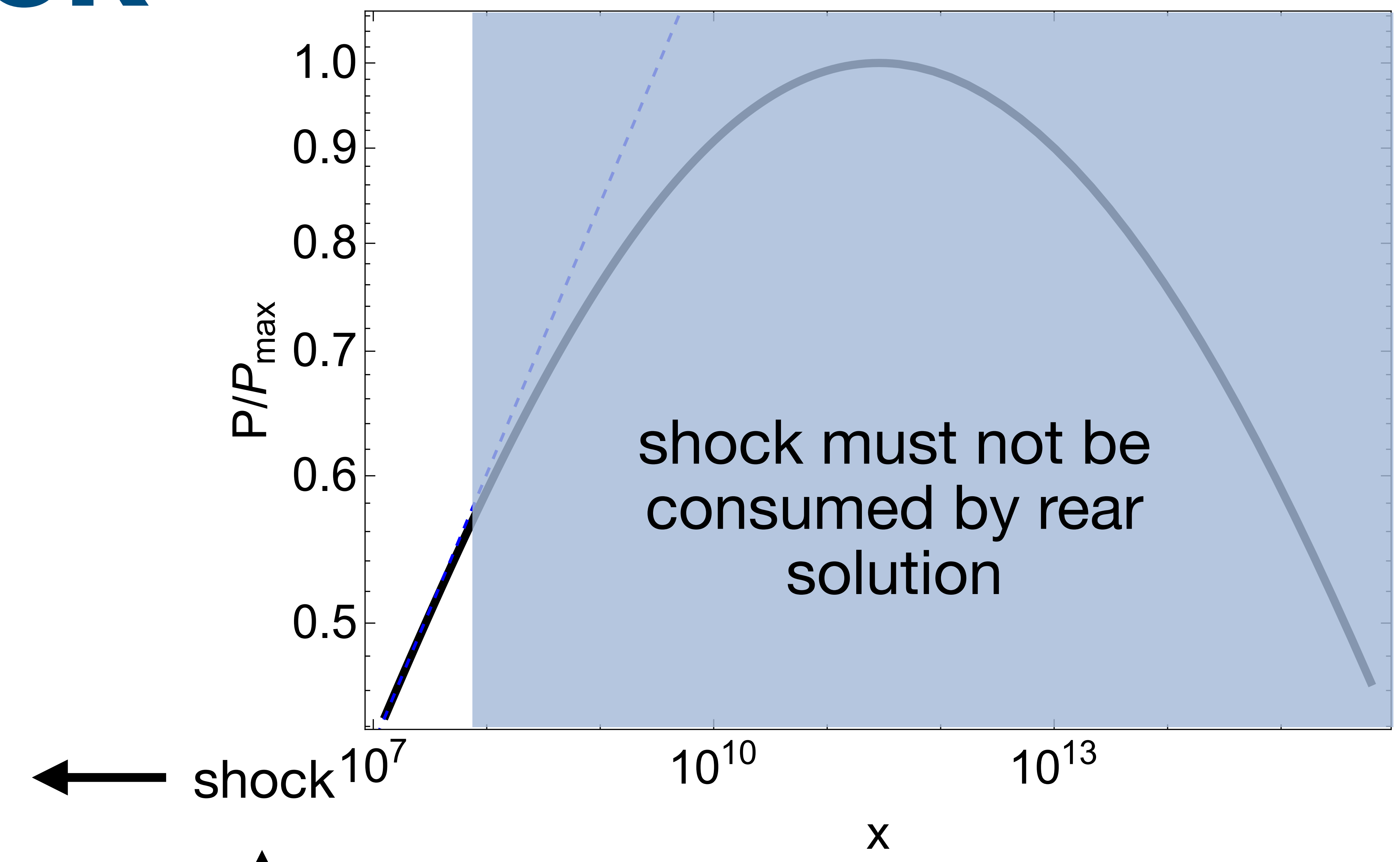

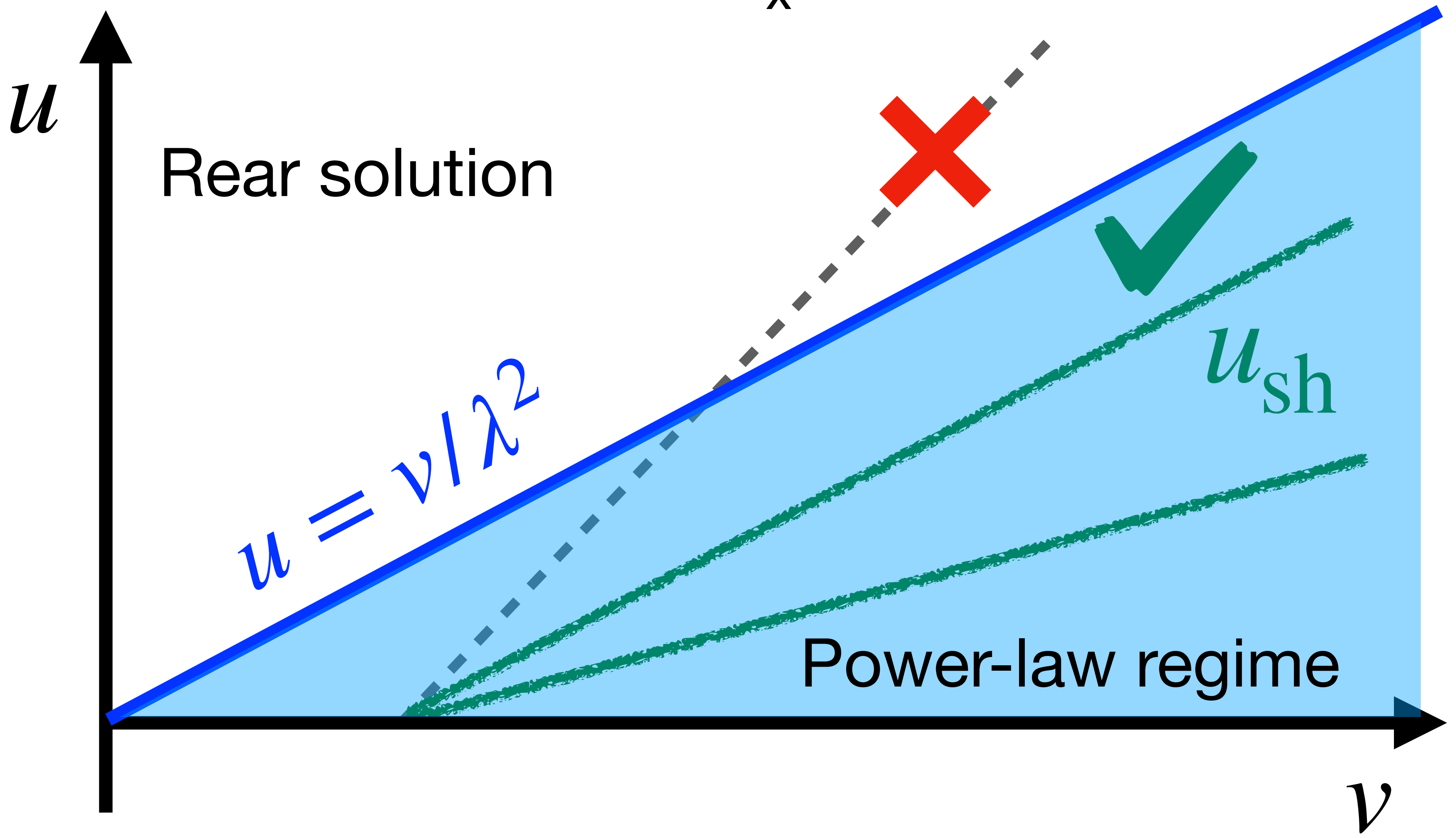

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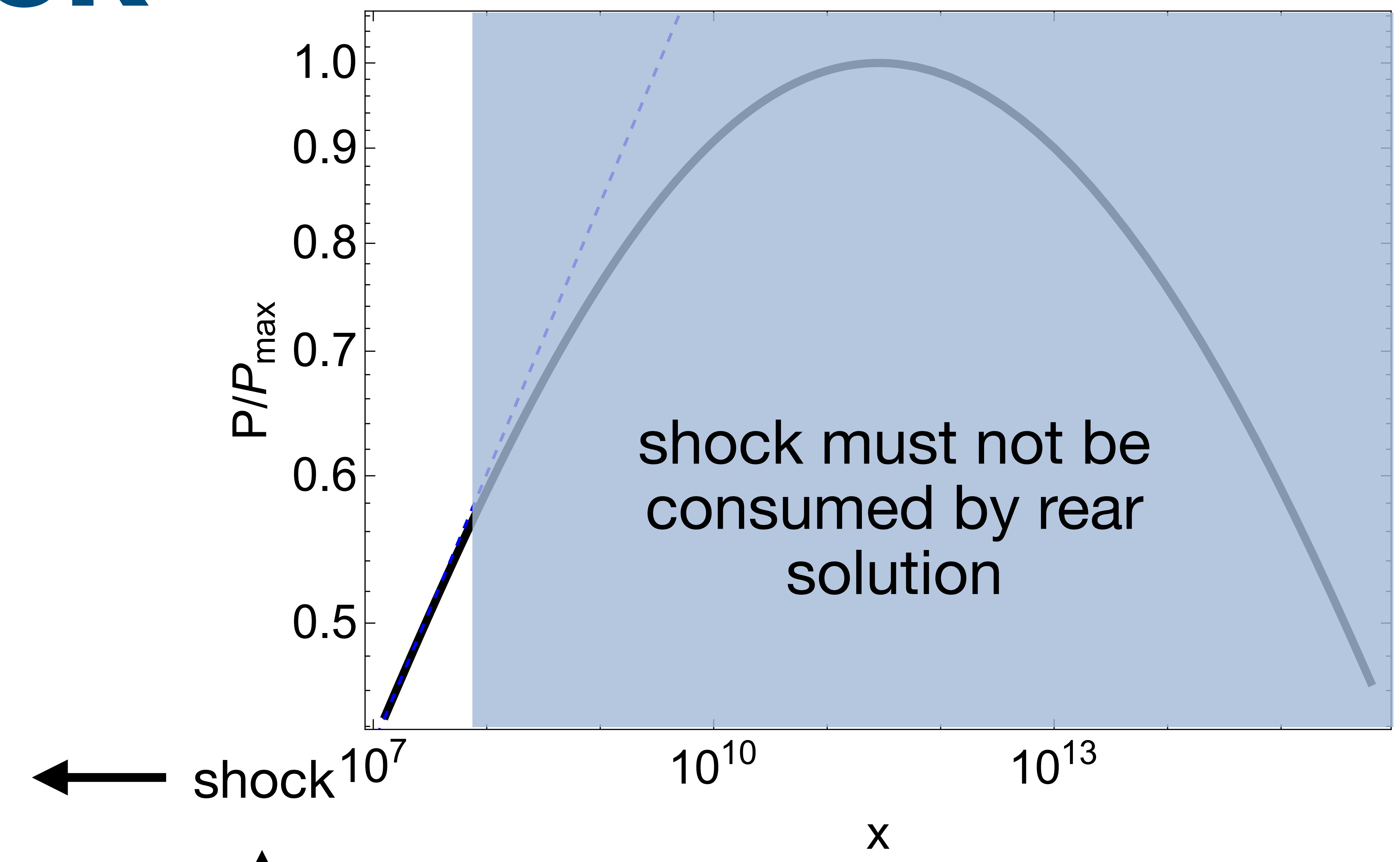
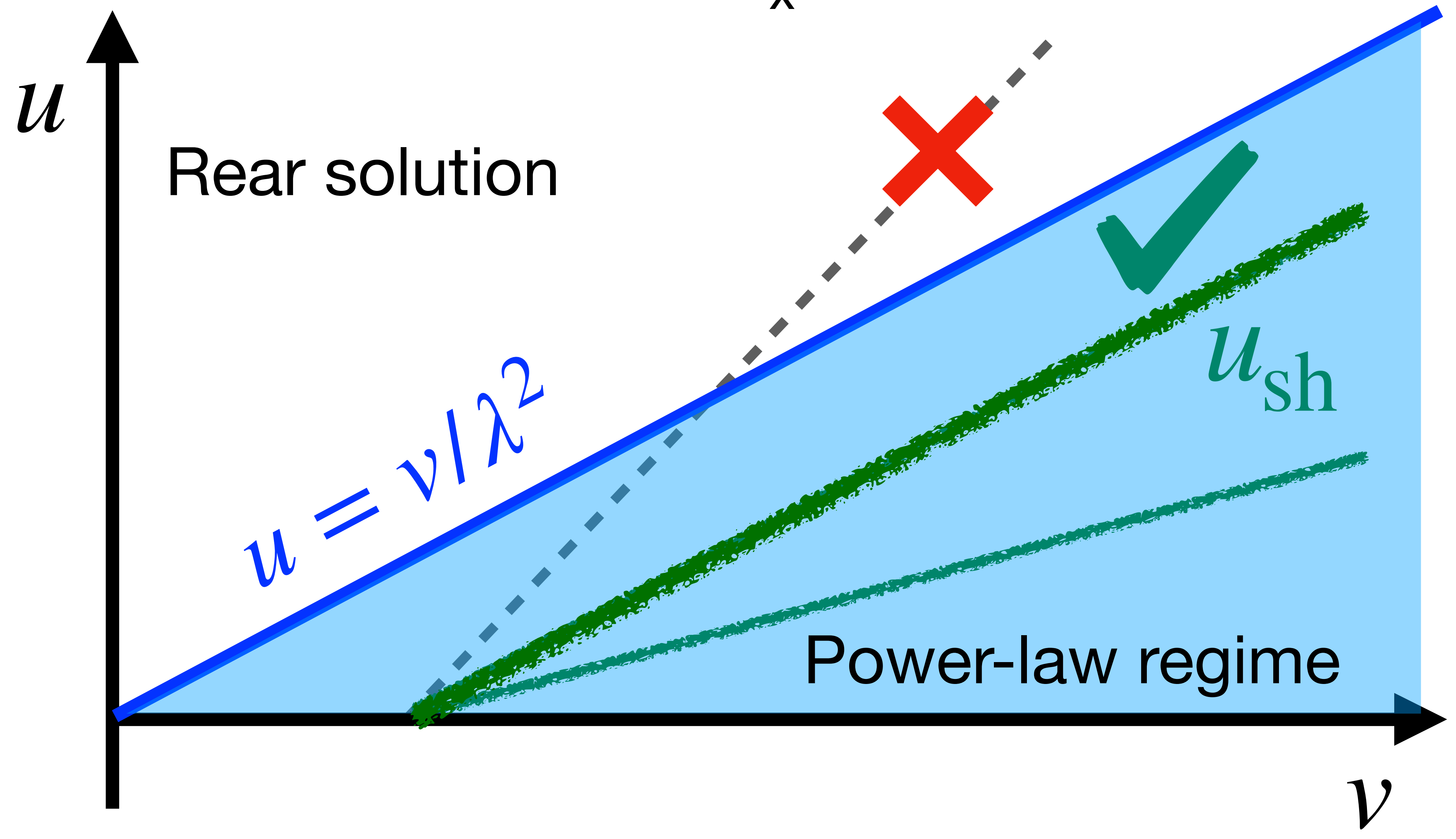


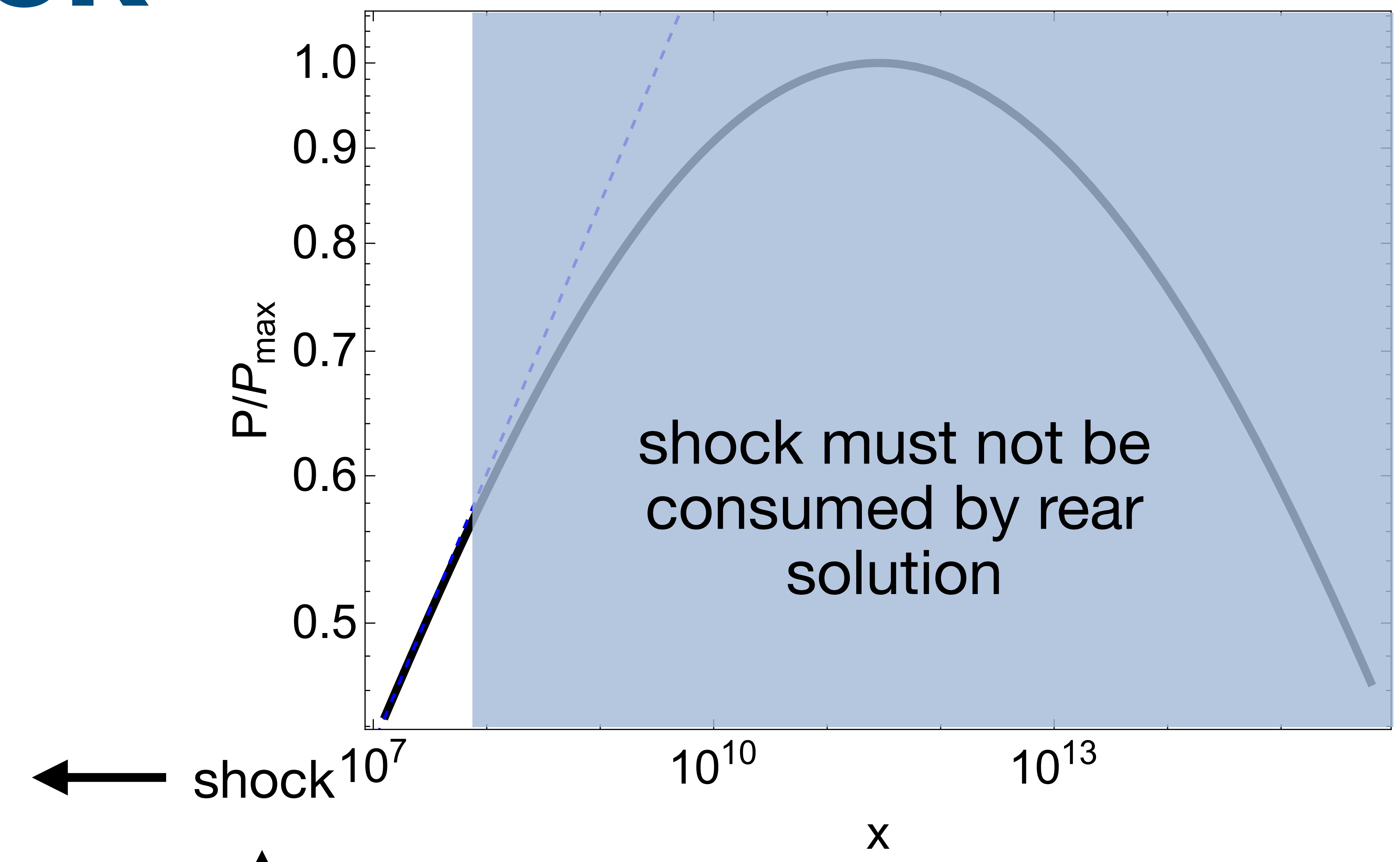


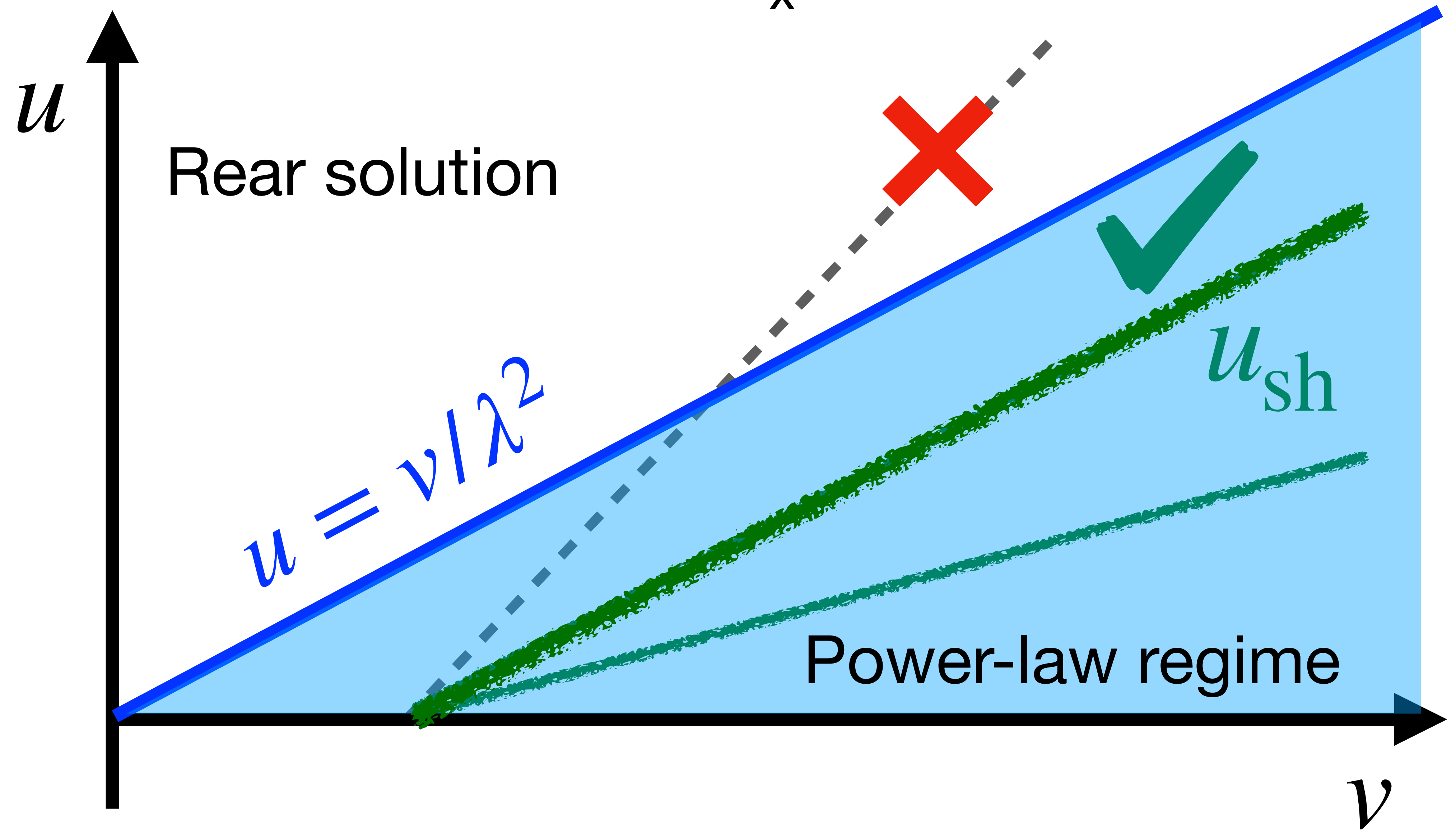
Consistency with a shock

What is the condition on $m(k)$?

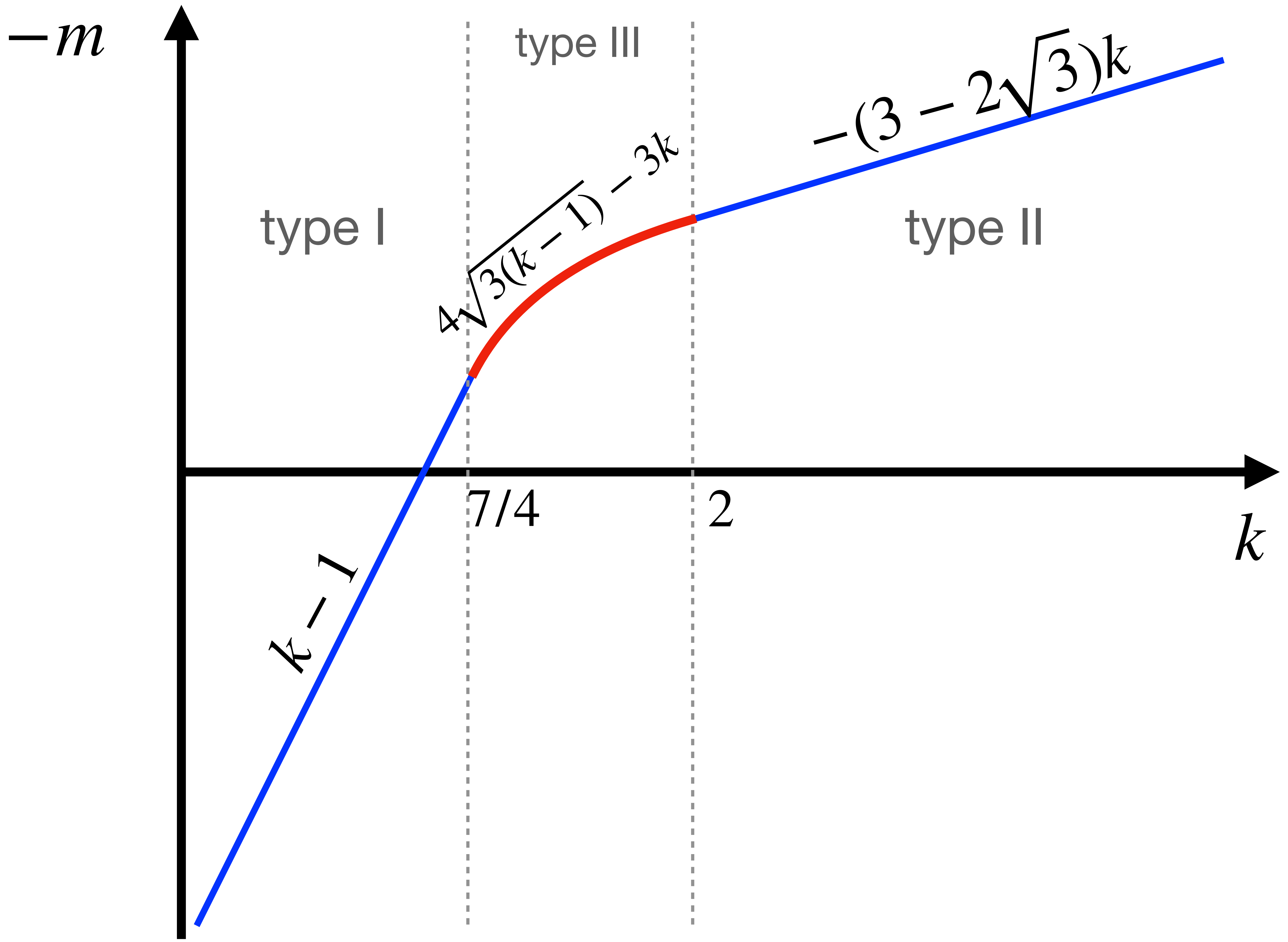
$$\left.\frac{du}{dv}\right|_{sh} = \frac{2m + \sqrt{3}(m+k)}{-2m + \sqrt{3}(m+k)} \leq \frac{1}{\lambda^2}$$

$$m_{\mathrm{III}} = -4\sqrt{3(k-1)} + 3k$$





The gap is closed



$\Gamma_{sh}^2(t) \propto t^{-m}$

$\rho_0 \propto r^{-k}$

Summary

(I) Refined classification:

Type I

Dimensional considerations

Type II

Eigenvalue problem within self-similar domain

Type III

Self-similar exponents determined by non self-similar flow

(II) New universal exponents for expansion into vacuum

$P_{\max} \propto t^{-1}$ $x_{\max} \propto t^{1/4}$ $\gamma^2(x_{\max}) \propto t^{3/4}$