A 4th-order accurate finite volume method for ideal and resistive classical and special relativisitc MHD in the PLUTO code

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Abstract

We present a novel implementation of a genuinely **4th-order accurate finite volume scheme** for multidimensional classical and special relativistic magnetohydrodynamics (MHD) in both ideal **[1]** and resistive **[2]** regimes based on the constrained transport (CT) formalism in the PLUTO code **[3]**. Our scheme is rooted over the method originally proposed by McCorquodale and Colella **[4]** but introducing several novel aspects when compared to its predecessors, yielding a **more efficient computational tool**. Among the most relevant ones, our scheme exploits **pointwise to pointwise reconstructions** (rather than one-dimensional finite volume ones), employs all the generalized upwind EMF averaging techniques of the UCT method of Mignone and Del Zanna **[5]**, with the addition of a **new relativistic UCT-GFORCE** average, and **ensures robustness** through sophisticated limiting strategies that include both a discontinuity detector and an order reduction procedure. Such method has so far produced results in numerical simulations **that are unfeasible with traditional low order schemes**.

The finite volume formalism in (relativistic) MHD

Our method is targeting the equations of ideal and resistive (R)MHD in Cartesian geometry.

1) **Hyperbolic sub-system** for the evolution of the conservative flow variables:

$$\frac{\partial \boldsymbol{U}}{\partial t} + \nabla \cdot \boldsymbol{F} = 0\,,$$

$$\boldsymbol{U} = \begin{pmatrix} \rho \\ \rho\vec{v} \\ \mathcal{E} \end{pmatrix},\quad \boldsymbol{F} = \begin{pmatrix} \rho\vec{v} \\ \rho\vec{v}\vec{v} - \vec{B}\vec{B} + \boldsymbol{I}p_t \\ (\mathcal{E}+p_t)\vec{v} - (\vec{v}\cdot\vec{B})\vec{B} \end{pmatrix}^{\intercal}$$

MHD: with ρ density, $\vec{v}$ fluid velocity, $\vec{B}$ magnetic field, p_t total pressure, and $\mathcal{E}$ energy density.

$$\boldsymbol{U} = \begin{pmatrix} D \\ \vec{m} \\ \mathcal{E} \end{pmatrix},\quad \boldsymbol{F} = \begin{pmatrix} D\vec{v} \\ w_t\gamma^2\vec{v}\vec{v} - \vec{B}\vec{B} - \vec{E}\vec{E} + \boldsymbol{I}p_t \\ \vec{m} \end{pmatrix}^{\intercal}$$

(Res)RMHD: with D relativistic mass density, $\vec{m}$ momentum density, p_t total pressure, and $\mathcal{E}$ energy density.

2) **Induction equation** for the evolution of the magnetic field:

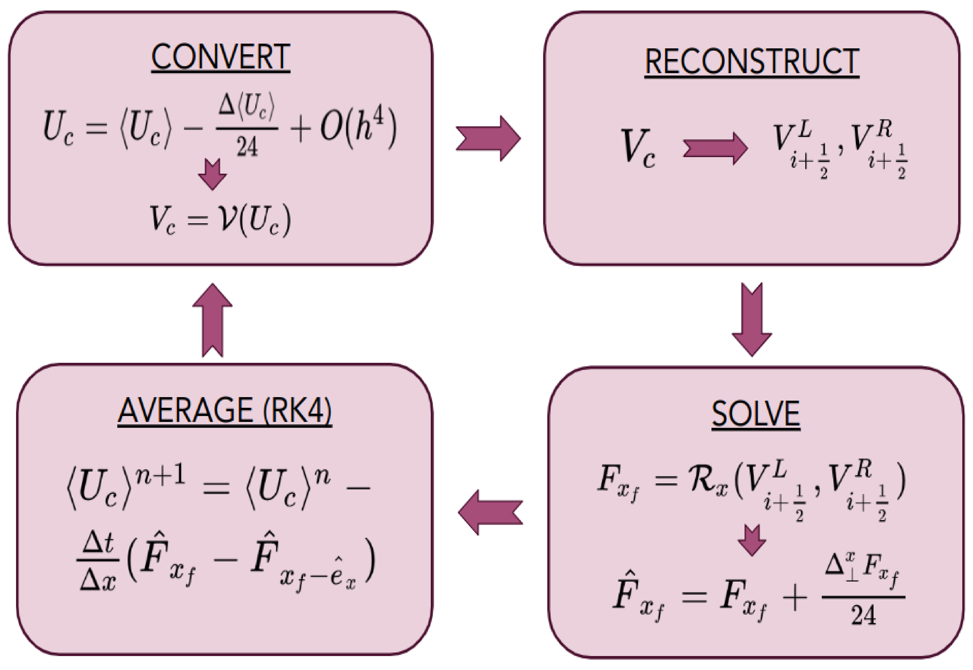
$$\frac{\partial \vec{B}}{\partial t} + c\nabla\times\vec{E} = 0.$$

endowed with the **solenoidal condition** $\nabla\cdot\vec{B} = 0$ satisfied at $t = 0$ and preserved $\forall t > 0$.

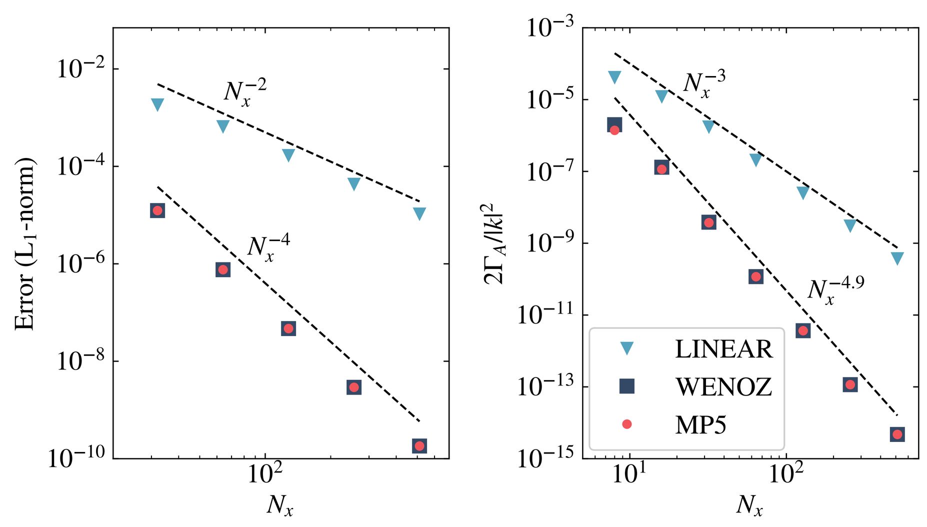
The 4th-order numerical scheme (in brief)

High order Godunov-type High-Resolution-Shock-Capturing scheme based on Reconstruct-Solve-Average strategy (for the details of the scheme see **[1]** and **[2]**):

1) **Conversion step** using Laplacian operators $\boldsymbol{\Delta}$ **[4]** $\Rightarrow$ obtain point values at cell centers $\boldsymbol{V_c}$.
2) **Pointwise Reconstructions** **[1]** reconstruct $\boldsymbol{V_c}$ at interfaces without having to average $\boldsymbol{V_c} \not\Rightarrow \langle\boldsymbol{V_c}\rangle$.
3) **Limiter** **[1]** introduction of 2 types of discontinuity detectors **[8]**,**[9]** and an order reduction procedure.
4) **Upwind Constrained Transport** + all the generalized upwind EMF averages **[5]** + punctual reconstructions.



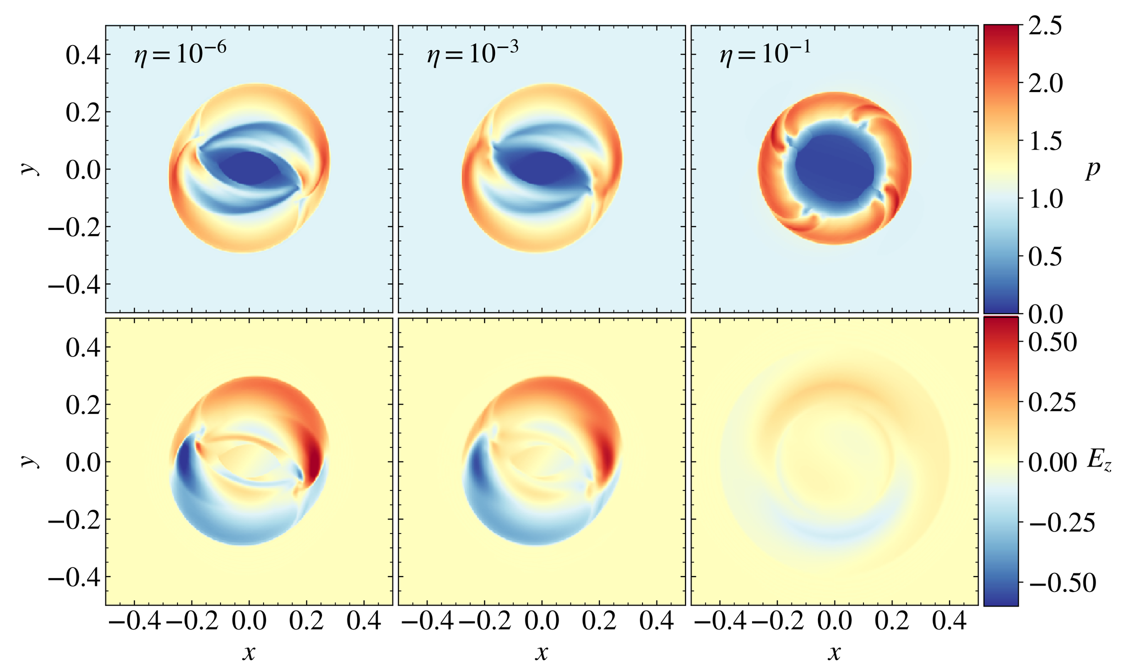
The 3D circularly polarized ideal Alfvén waves



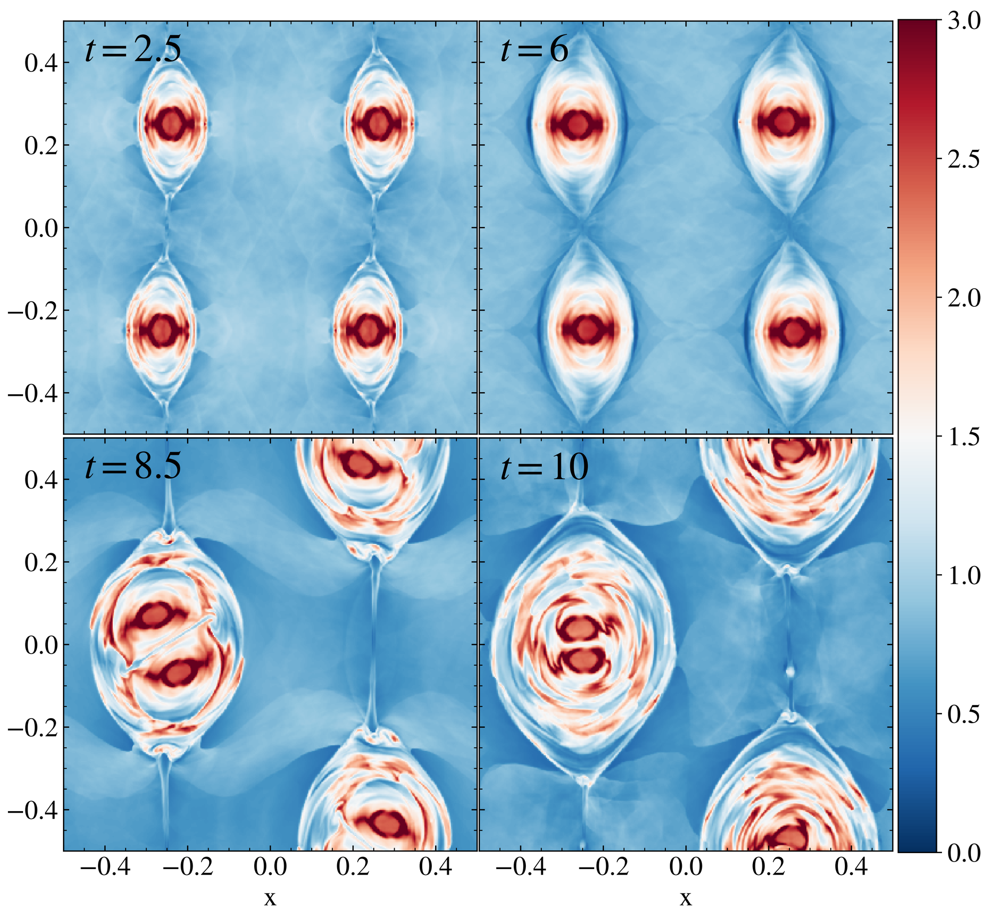
- Convergence rates match the expected order $\Rightarrow$ **improved accuracy**.
- For fixed accuracy $\epsilon_p = CN^{-p}$ (p = 2, 4) 2nd- and 4th-order scheme will achieve $\epsilon_2 \sim \epsilon_4$ with $N_4 \sim \sqrt{N_2}$ grid points.
- Net gain at fixed accuracy estimated: ϵ_2 at $N_2^3 = 512^3 \simeq \epsilon_4$ at $N_4^3 = 34^3$ + computing time reduced by a factor $10^4 \Rightarrow$ **saving of computational time**.
- More accurate scheme $\Rightarrow$ **reduced numerical dissipation**.
- Overall decay rate of $\sim N^{-4.9}$ for the 4th-order scheme versus only $\sim N^{-3.0}$.

The ResRMHD rotor

- High density disk rotating at relativistic speed ($\omega = 8.5$) simulated with increasing resistivity: $\eta = 10^{-6}, 10^{-3}, 10^{-1}$.
- The complex pattern of shocks and torsional Alfvén waves is correctly reproduced by the new scheme.
- Magnetic braking slows down the rotor and **physical resistivity** dissipates the electric field.



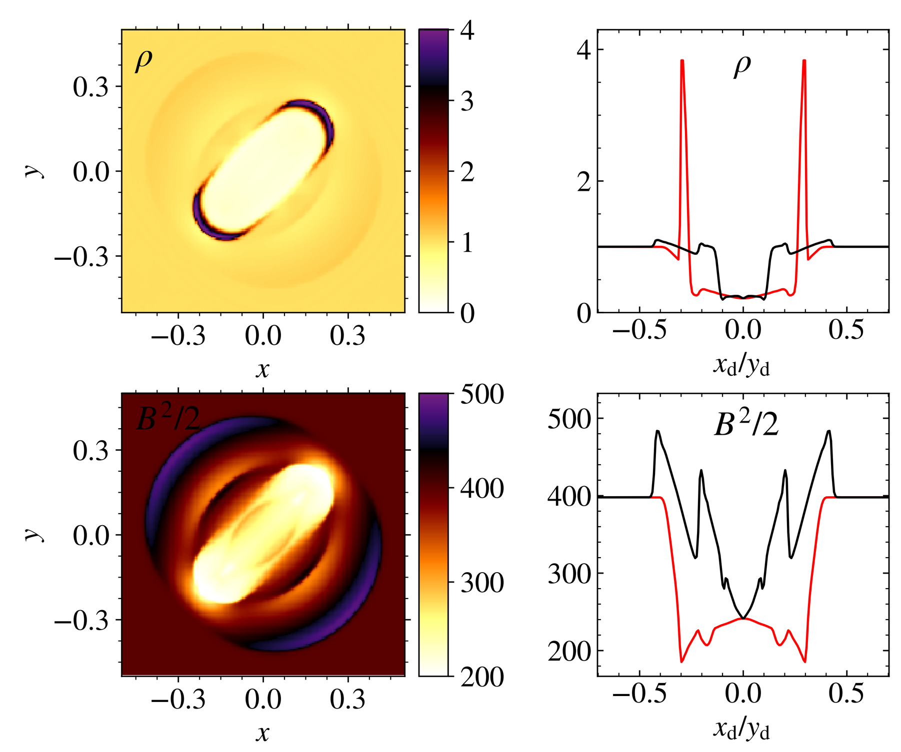
The relativistic current sheet



- Upwind Constrained Transport method using all the EMF averages of **[5]** to ensure $\nabla\cdot\vec{B} = 0$.
- Pointwise reconstructions reduce the number of transverse ghost zones to reconstruct the electric field $\Rightarrow$ **more efficient CT algorithm**.
- Extended the MHD **UCT-GFORCE** average to RMHD $\Rightarrow$ accurate and robust solver.
- This ideal relativistic current sheet reproduces the one by **[6]**, but with a more accurate and less dissipative solver (GFORCE rather than linear solver).

The 3D MHD blast wave

- Severe blast configuration **[7]** with $B_0 = 100/\sqrt{4\pi}$, $p_{out} = 1$ and $p_{in} = 10^3$.
- So far only feasible with 2nd-order schemes.
- Limiter acts in two stages of the algorithm:
  1) during the conversion $\langle \boldsymbol{U_c}\rangle \Rightarrow \boldsymbol{U_c}$ for **reverting to 2nd-order** with Jameson's shock sensor **[8]** or by a more sensitive limiter based on higher order derivatives' ratio **[9]**;
  2) during reconstruction at interfaces by means of an **order reduction** procedure when oscillations are generated.



Results, current and future development

- Genuinely **4th**-order accurate finite volume scheme for both ideal and resistive (R)MHD accounting several innovative aspects that yield an accurate, robust and efficient computational tool;
- Introduction of pointwise to pointwise reconstruction operations that ease up the structure of the scheme;
- Robustness assured by a limiter that allows the **4th**-order scheme to carry out extremely severe tests yielding unprecedented results;
- Generalization of the UCT-GFORCE average to relativistic MHD;
- Extension to non-Cartesian geometries (A. Mignone & **V. Berta**, in prep.);
- Extension to non-ideal frameworks: resistive RMHD module (A. Mignone, M. Rossazza, **V. Berta** et al. in prep.) to study energy dissipation and plasmoid formation in 3D plasma columns;
- Extension and applications of the **4th**-order dust fluid module to the streaming instability (ongoing with the UFOs group at the MPIA, Heidelberg, Germany);
- Extension of the **4th**-order method to GRMHD;

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